Sep 17: Volume Element, Differences between Connection and Tensor, Parallel Transport

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1 Volume

We can write:

$$d^n x \to dx^0 \wedge .. \wedge dx^{n-1} \tag{1}$$

So that the volume element can be defined as:

$$dV = \sqrt{-g} d^n x = dV' = \sqrt{-g'} d^n x' \tag{2}$$

Above, the $\sqrt{-g}$ takes the "role" of the $\wedge$'s in (1). We define the volume element as an n-form by

$$\epsilon = \epsilon_{\mu_1 ... \mu_n} dx^{\mu_1} \otimes ... \otimes dx^{\mu_n} = \frac{1}{n!} \tilde{\epsilon}_{\mu_1 ... \mu_n} dx^{\mu'} \wedge ... \wedge dx^{\mu_n} \tag{3}$$

Where above the $\otimes$ means tensor product. We provide a proof of how the dV changes to dV', but it uses the fact that:

$$dx^{\mu_1} \wedge ... \wedge dx^{\mu_n} = \left| \frac{\partial x^\mu}{\partial x^{n\prime}} \right| dx^{\mu_1'} \wedge .. \wedge dx^{\mu_n'} \tag{4}$$

and the fact that,

$$\sqrt{-g} \left| \frac{\partial x^\mu}{\partial x^{n\prime}} \right| = \sqrt{-g'} \tag{5}$$

2 Curvature

It is important to note that the Christoffel symbol Γ is NOT a tensor, for the simple reason that it does not transform like a tensor. Only special combinations of the Christoffel Symbols yield a tensor object. So we call the Christoffel symbols connections.

Theorem if $\exists$ $C^\nu_{\mu\sigma}$ such that $\partial_\mu V^\nu + C^\nu_{\mu\sigma} V^\sigma$ transforms as a tensor (i.e. $\nabla_\mu V^\nu$) then $C^\nu_{\mu\sigma}$ is a connection.

To make this clear, a tensor transforms like:

$$\partial_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu\prime}} \frac{\partial x^{\nu\prime}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu\prime}} \frac{\partial x^\sigma}{\partial x^{\sigma\prime}} \frac{\partial^2 x^{\nu\prime}}{\partial x^\nu \partial x^\sigma} V^{\sigma\prime} \tag{6}$$

but a connection transforms like this:

$$C^{\nu'}_{\mu'\sigma'} = \frac{\partial x^{\nu\prime}}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^{\mu\prime}} \frac{\partial x^\sigma}{\partial x^{\sigma\prime}} C^\nu_{\mu\sigma} + \frac{d^2 x^{\nu\prime}}{\partial x^\mu \partial x^\sigma} \frac{\partial x^\mu}{\partial x^{\mu\prime}} \frac{\partial x^\sigma}{\partial x^{\sigma\prime}} \tag{7}$$

which shows these objects transform in different ways.

We mentioned we can define a tensor via a special combination of these connections. For example, for two connections $C^\nu_{\mu\sigma}$ and $C^\nu_{\mu\sigma}$ we can define a tensor $S^\nu_{\mu\sigma}$ as: $S^\nu_{\mu\sigma} = C^\nu_{\mu\sigma} - \tilde{C}^\nu_{\mu\sigma}$. This object on the left hand side is a tensor, even though the individual objects on the right hand side are connections. An example would be if C and $\tilde{C}$ where the christoffel symbols evaluated at x^μ

and $x^\mu + \delta x^\mu$ respectively. This leads us to conclude that the object, $d(\Gamma^\nu_{\mu\sigma}) = \Lambda^\nu_{\mu\sigma\tau} dx^\tau$ may be treated as a tensor.

For example, the Riemann Curvature tensor looks like:

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{8}$$

The objects on the right hand side are not by themselves tensors, the special combination written above makes the left hand side a, anti-symmetric in ν, tensor. That is, the whole thing behaves as a tensor. For a general $C^\mu_{\nu\sigma}, \nabla_\mu g_{\tau\sigma} \neq 0$. Also, we can only act the covariant derivative ∇_τ on tensors, so something like $\nabla_\tau \Gamma$ is nonsense.

We also define a torsion tensor, although of so far little to no observational evidence, as $T^\nu_{\mu\sigma} = C^\nu_{\mu\sigma} - C^\nu_{\sigma\mu}$. In order for us to define the Christoffel symbols as "metric compatible", we must require the following conditions:

$$i)\; T^\mu_{\nu\sigma} = 0, \tag{9}$$

$$ii)\; \nabla_\mu g_{\tau\sigma} = 0 \quad \forall \mu, \sigma, \tau \tag{10}$$

3 Parallel Transport

Vectors don't live in the Manifold M, they live in the tangent space at a point P on M. And so we define parallel transport as, for a vector V^μ along a curve $\frac{d}{d\lambda}$, $\frac{D}{d\lambda}(V^\mu) = 0$. So the closest we can get, to having this vector not change is:

$$\frac{dx^\nu}{d\lambda} \nabla_\nu V^\mu = 0 \tag{11}$$

For example, the usual acceleration: $a^\mu = \frac{d}{d\tau} V^\mu = \frac{dx^\nu}{d\tau} \nabla_\nu V^\mu$

Not every tensor can be parallely transported, only those that satisfy the parallel transport equation above. However if $\nabla_\sigma g_{\mu\nu} = 0$, then we can say that the metric tensor can be parallely transported along any curve.