General Relativity

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September 14,2018

1 Tensor Densities

The Levi-Civita symbol $\tilde{\epsilon}$ is an antisymmetric nontensorial object which has the components specified below *in any* right-handed *coordinate system*

Levi-Civita is not a tensor because it does not change under a coordinate transformation.

$$\tilde{\epsilon}_{\mu_1 ... \mu_n} \neq \tilde{\epsilon}_{\mu_1' ... \mu_n'} \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} ... \frac{\partial x^{\mu_n'}}{\partial x^{\mu_n}}$$

The Kronecker delta $\delta^{\mu}{}_{\nu} = \text{Diag}(1...1)$ is a (1,1) tensor and so it transforms like

$$\delta^{\mu'}{}_{\nu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \delta^{\mu}{}_{\nu} \frac{\partial x^{\nu}}{\partial x^{\nu'}} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} = \delta^{\mu'}{}_{\nu'}$$

Consider the determinant of an $n \times n$ matrix $M^{\mu}{}_{\mu'}$ defined

$$M^{\mu}{}_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}$$

The determinant is

$$\epsilon_{\mu_1' ... \mu_n'} |M| = M^{\mu_1}{}_{\mu_1'} ... M^{\mu_n}{}_{\mu_n'} \tilde{\epsilon}_{\mu_1 ... \mu_n}$$

$$\tilde{\epsilon}_{\mu_1' ... \mu_n'} = |M|^{-1} M^{\mu_1}{}_{\mu_1'} ... M^{\mu_n}{}_{\mu_n'} \tilde{\epsilon}_{\mu_1 ... \mu_n} = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right| \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} ... \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}} \tilde{\epsilon}_{\mu_1 ... \mu_n}$$

Consider the scalar $|g_{\mu\nu}| = g$. If it's not a matrix, how can we find its determinant? To convert to a tensor, we can multiply by $|g|^{w/2}$ where w is the weight of the density. The absolute sign exists because Lorentzian metrics have $g < 0$.

$$\text{Det} g = \frac{1}{n!} g_{\mu_1 \nu_1} ... g_{\nu_n \mu_n} \tilde{\epsilon}^{\mu_1 ... \mu_n} \tilde{\epsilon}^{\nu_1 ... \nu_n}$$

$$g' = |g_{\mu'\nu'}| = \left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} g_{\mu\nu} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \right| = \left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right|^2 |g_{\mu\nu}| = \left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right|^2 g = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right|^{-2} g$$

$$\sqrt{-g'} \tilde{\epsilon}_{\mu_1' ... \mu_n'} = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right| \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right|^{-1} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} ... \frac{\partial x^{\mu_n}}{\partial x^{\mu_n}} \sqrt{g} \epsilon_{\mu_1 ... \mu_n} = \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} ... \frac{\partial x^{\mu_n}}{\partial x^{\mu_n}} \sqrt{g} \epsilon_{\mu_1 ... \mu_n}$$

Now g is a not a scalar, but a (scalar) density of weight -2, which is the power to which the Jacobian is raised. The Levi-Civita symbol is a density of weight 1.

So what we say is that $\tilde{\epsilon}$ is a tensor density of rank 0.

$\sqrt{-g}\tilde{\epsilon}_{\mu_1\ldots\mu_n}$ is a tensor $= \epsilon_{\mu_1\ldots\mu_n}$.
The difference comes up because it is not a tensor, it's the current density. The way we usually use the current is a current density instead of an integrated current. It turns out that in the Hamiltonian formulation, one uses the spatial components of g_{ij} as the canonical variables and the tensor density is the momentum conjugate Π^{ij}.

$g_{\sigma\sigma}$ and $g_{\sigma i}$ will both turn out to be Lagrange multipliers, which means they are not dynamical variables. Tensor densities play an important part in Hamiltonian theory.

Next topic is something that seems unrelated but will become more relevant before the end of the day.

2 (Differential) Forms

A scalar has no indices so it is called a scalar form. The gradient of a function has one index and in our language it will fit into the classifications of a one-form, previously treated as a covector.

$$\begin{aligned} \phi &- \text{ 0-form (scalar)} \\ \nabla_\mu f &- \text{ 1-form (covector, for example the vector potential } A_\mu) \\ F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu &- \text{ 2-form (totally antisymmetric)} \end{aligned}$$

k-forms are a subclass of $(0,k)$ tensors.

Suppose we have a p-form $\mathbb{A}$ and a q-form $\mathbb{B}$. Then we denote the $(p+q)$-form $(\mathbb{A}\wedge\mathbb{B})_{p+q}$,

$$(\mathbb{A}\wedge\mathbb{B})_{p+q} = \frac{(p+q)}{p!q!}(A_{[\mu_1\ldots\mu_p}B_{\mu_{p+1}\ldots\mu_{p+q}]})$$

Consider:

- if $\mathbb{A}$, $\mathbb{B}$ are 1-forms,

$$(\mathbb{A}\wedge\mathbb{B}) = \frac{2!}{1!1!}\cdot\frac{1}{2}(A_\mu B_\nu - A_\nu B\mu)$$

- if $\mathbb{A}$ is a 1-form (A_μ) and $\mathbb{B}$ is a 2-form ($B_{\mu\nu} = -B_{\nu\mu}$).

$$\begin{aligned} C_{\mu\nu\sigma} = (\mathbb{A}\wedge\mathbb{B})_{\mu\nu\sigma} &= \frac{3!}{1!2!}\frac{1}{6}(A_\mu B_{\nu\sigma} + A_\nu B_{\sigma\mu} + A_\sigma B_{\mu\nu} - A_\mu B_{\sigma\nu} - A_\nu B_{\mu\sigma} - A_\sigma B_{\nu\mu}) \\ &= (A_\mu B_{\nu\sigma} + A_\nu B_{\sigma\mu} + A_\sigma B_{\mu\nu}) \\ C_{\nu\mu\sigma} &= (A_\nu B_{\mu\sigma} + A_\sigma B_{\nu\mu} + A_\mu B_{\sigma\nu}) = -C_{\mu\nu\sigma} \end{aligned}$$

We could have a basis of many different forms. Right now, we haven't specified anywhere a dimensionality. We could be working in 3D, 4D, or even 2D (although this wouldn't work in 2D so we must work in at least 3D).

Hodge Duality

There exists a dual space for a k-form

$$(*A)_{n-k} = \epsilon^{\mu_1 \ldots \mu_k}{}_{\nu_{k+1 \ldots n}} A_{\mu_1 \ldots \mu_k}$$

The above is an $(n-k)$-form. The *Hodge operator*$(*)$ is a map from k-forms to (n-k)-forms, mapping A to 'A dual' (Hodge dual). So we can map back and forth between them

$$(**A) = (-1)^{s+k(n-k)} A \qquad s = \begin{bmatrix} -+++ = 1 \\ +--- = 3 \end{bmatrix}$$

In 4D, the dual $(*F)$ is a 2-form.

$$F = \begin{bmatrix} 0 & & & F_{ij} \\ & 0 & & \\ & & 0 & \\ -F_{ij} & & & 0 \end{bmatrix} \qquad \nabla_\mu F^{\mu\nu} = 0$$

In 3D, the dual of the wedge of two 1-forms is a 1-form. This is also why $A \times B$ is a pseudo-vector. It has three components but does not transform as a vector, where the appearance of the Levi-Civita tensor explains why the the cross product changes sign under interchange of two basis vectors.

If A is a p-form we can look at

$$(\mathrm{d}A)_{p+1} \quad \text{form} = (p+1)\partial_{\mu_1 \ldots \mu_{p+1}}$$

The d(dA) of anything is zero.