Special and General Relativity-I, Fall 2018

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September 10, 2018

Vectors and Tangent Spaces

Before we proceed to study the above mentioned structures, we recall the following definitions:

- **Maps:** Let M and N be two non-empty sets. A map $\phi : M \longrightarrow N$ is a subset, U, of $M \times N$ such that for every $m \in M$ there is a unique $n \in N$ such that the ordered pair (m, n) is in U. Loosely speaking, a map is a "rule" that associates with each element m in M a unique element n in N.
  A map $\phi : M \longrightarrow N$ is called onto if given $n \in N \exists m \in M : \phi(m) = n$.
  A map $\phi : M \longrightarrow N$ is called one-to-one if each element of N has at most one element of M mapped into it, that is, if whenever $m_1 \neq m_2$, then $\phi(m_1) \neq \phi(m_2)$.

- **Open set:** A set is called open if all of its elements (points) have at least one neighborhood which completely lies in the set. Eg, an open interval (a, b) in $\mathbb{R}$.

- **Chart:** A chart consists a subset U of a set M, along with a one-to-one map $\phi : U \longrightarrow \mathbb{R}^n$ such that the image $\phi(U)$ is open in $\mathbb{R}^n$. Being injective, ϕ is invertible in the region $\phi(U) \subset \mathbb{R}^n$ so we have an isomorphism between U and a region of $\mathbb{R}^n$.

- **Atlas:** An Atlas is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ that satisfies the following conditions:

  1 $\cup U_\alpha = M$, that is, charts are like patches and with these patches we can cover all of the space, when taken together. We can think of this as trying to cover an irregular portion of earth having hills and valleys with carpets. We cannot cover the whole region by a single carpet but instead we can use several small carpets to cover the area. Compare this Atlas with the Atlas used in secondary level school which contained a set of maps of various portion of the earth. This is a very good way of describing $2-D$ space without embedding earth in $3-D$.

  2 In the above example of carpets, we also want the carpets to overlap smoothly if they overlap. This is guaranteed by this condition.
  If two charts overlap, $U_\alpha \cap U_\beta \neq \{\}$, then the map$(\phi_\alpha \circ \phi_\beta)$ takes a point in $\phi_\beta(U_\alpha \cap U_\beta)$ onto an open set $\phi_\alpha(U_\alpha \cap U_\beta)$ and these maps should be as many times differentiable as we want, that is, the maps should be smooth.

- **Manifold:** A manifold is a set M along with an atlas which contains all the charts that are compatible with each other.

Now that we have a manifold we want to define vectors in it. Vector should not be considered as something that stretches from one point of a maifold to another point, but instead is just an object associated with a single point of the manifold. In order to generalize the idea of vector from ordinary Euclidean spaces to general manifolds we discuss tangent spaces, because they are the spaces of the vectors associated with each point of a manifold. And, in defining tangent space, we want an intrinsic definition without the need to embed our manifold in higher dimensional space.

The first intuitive idea is obviously taking all the curves at a point in the manifold and defining a vector space of the tangent vectors to the curves at the point. To do so, we define $\mathcal{F}$ to be the space of all smooth function on M. Then, along each curve $\gamma : \mathbb{R} \longrightarrow M$, we can define an operator, the directional derivative, on this space, which maps $f \longrightarrow df/d\lambda$, where λ is the parameter of the curve. Then, we can define the tangent space T_p to be the space of directional derivative operators along the curves through p.

To establish this definition, we need to show that the space of derivative operator is indeed a vector space. The addition and multiplication by a scalar is straightforward and the properties of a vector space are satisfied in a straightforward way. Secondly, we need to show that it's dimension is same as the dimension of our manifold so that we can claim that it is the space we desired to have. For this we consider
a chart $\phi : M \longmapsto \mathbb{R}^n$,
a curve $\gamma : \mathbb{R} \longmapsto M$, λ being parameter

and a function $f : M \longmapsto \mathbb{R}$. Also, look at the attached picture to understand the following steps.

$$\begin{aligned}
\frac{df}{d\lambda} &= \frac{d(f\circ\gamma)}{d\lambda} \\
&= \frac{d(f\circ\phi^{-1})\circ(\phi\circ\gamma)}{d\lambda} \\
&= \frac{d(\phi\circ\gamma)^\mu}{d\lambda}\circ\frac{\partial(f\circ\phi^{-1})}{\partial x^\mu}, \text{ chain rule, see (2.12), Carroll} \\
&= \frac{dx^\mu}{d\lambda}\partial_\mu f \\
\text{or, } \frac{d}{d\lambda} &= \frac{dx^\mu}{d\lambda}\partial_\mu
\end{aligned}$$

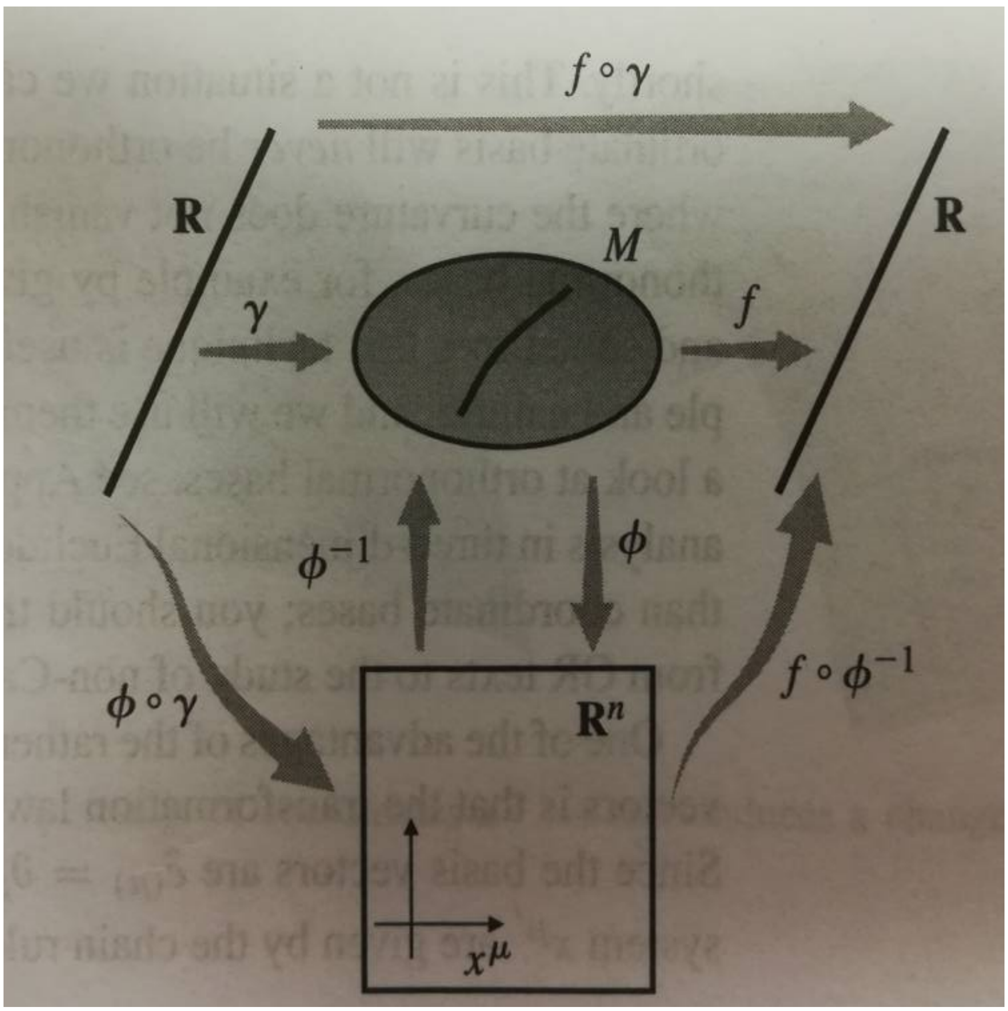

Figure 1: Decomposing a tangent vector to a curve γ in terms of partial derivatives wrt coordinates on M. Source: Carroll, Spacetime and Geometry

Thus, the partials $\{\partial_\mu\}$ is a basis for the vector space of directional derivatives, which means the dimension of this space is same as the dimension of the manifold and we call this space the tangent space T_p. The vector represented by $d/d\lambda$ is the tangent vector, with components $dx^\mu/d\lambda$ with respect to the basis $\{\partial_\mu\}$. This basis is called coordinate basis.

As we have a vector in our hand now, we want to change coordinates from x^μ to $x^{\mu'}$, then,

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu$$

and, for a vector V, which as a whole shouldn't change,

$$\begin{aligned} V^\mu \partial_\mu &= V^{\mu'}\partial_{\mu'} \\ &= V^{\mu'}\frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu \end{aligned}$$

so that (prime $\longleftrightarrow$ unprime) gives

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}V^\mu \tag{1}$$

This is the transformation law of vectors.

A generic vector in tangent space can be written as

$$\mathbb{X} = X^\mu \partial_\mu$$

If we have two such vectors then

$$\begin{aligned} \mathbb{X}\mathbb{Y} &= X^\nu\partial_\nu Y^\mu\partial_\mu = X^\nu Y^\mu \partial_\nu\partial_\mu + X^\nu \partial_\nu Y^\mu \partial_\mu \\ \mathbb{Y}\mathbb{X} &= Y^\nu\partial_\nu X^\mu\partial_\mu = Y^\nu X^\mu \partial_\nu\partial_\mu + Y^\nu \partial_\nu X^\mu \partial_\mu \\ \text{so that,} \quad \mathbb{X}\mathbb{Y} - \mathbb{Y}\mathbb{X} &= \left(X^\nu\partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu\right)\partial_\mu \end{aligned}$$

so the commutator defines a vector

$$[\mathbb{X}, \mathbb{Y}] = Z^\mu \partial_\mu \tag{2}$$

where,

$$Z^\mu = \left(X^\nu\partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu\right)$$

We might suspect that this is not a vector because a partial derivative $\partial_\mu V^\nu$ is not a tensor, in general. Using the definition of Covariant derivatives using Christoffel Symbol,

$$\begin{aligned} Z^\mu &= X^\nu\partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu \\ &= X^\nu\left(\nabla_\nu Y^\mu - \Gamma^\mu_{\sigma\nu}Y^\sigma\right) - Y^\nu\left(\nabla_\nu X^\mu - \Gamma^\mu_{\sigma\nu}X^\sigma\right) \\ &= X^\nu\nabla_\nu Y^\mu - Y^\nu\nabla_\nu X^\mu \end{aligned}$$

Therefore, we see that the non-tensor christoffel symbol cancels out and we get a pure vector.

In flat space-time,

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

so the metric tensor $g_{\mu\nu}$ is independent of (t, x, y, x), which means the following vectors commute with the metric.

$$\begin{aligned} \mathbf{t} = 1.\partial_t &\Longrightarrow t^\mu = (1,0,0,0) \\ \mathbf{x} = 1.\partial_x &\Longrightarrow x^\mu = (0,1,0,0) \\ \mathbf{y} = 1.\partial_y &\Longrightarrow y^\mu = (0,0,1,0) \\ \mathbf{z} = 1.\partial_z &\Longrightarrow z^\mu = (0,0,0,1) \end{aligned}$$

However, in a different coordinate system, things might be different. Consider spherical polar coordinate system,

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$$

Here, the metric depends on (r,θ). Then if

$$\phi = 1.\partial_\phi \Longrightarrow \phi^\mu = (0,0,0,1)$$

which gives, in this coordinates,

$$\phi.\mathbf{g}.\phi = r^2 \sin^2\theta$$
$$\mathbf{t}.\mathbf{g}.\mathbf{t} = -c^2$$

To give an idea of choices of different basis, lets define another vector,

$$\psi = \frac{1}{r\sin\theta}\partial_\phi$$

The usual basis which we used to take in EM would be $(\partial_t, \partial_r, \frac{1}{r}\partial_\theta, \frac{1}{r\sin\theta}\partial_\phi)$ so the components of ψ are $(0,0,0,1)$. The another choice of basis, which we usually take here, is $(\partial_t, \partial_r, \partial_\theta, \partial_\phi)$ and the components would be $(0,0,0,1/r\sin\theta)$.

Cotangent Vectors

Cotangent Space T_p^* is a vector space of functionals $\omega : T_p \longmapsto \mathbb{R}$. The action of a cotangent vector upon a vector should therefore be a scalar. If we take a vector and an one-form

$$\mathbb{V} = \frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu$$
$$\mathbb{W} = df = \partial_\mu dx^\mu, \quad \text{so,}$$
$$\mathbb{W}.\mathbb{V} = \frac{dx^\mu}{d\lambda}\partial_\mu f$$

which is a scalar and should not be confused with the vector

$$\left(\frac{dx^\mu}{d\lambda}\partial_\mu\right) f$$

Just as partial derivatives along coordinate axes give us a basis for the tangent space, the gradients of the coordinate functions give a basis for the cotangent space.

$$dx^\mu \partial_\nu = \delta^\mu_\nu$$

Therefore, the gradients $\{dx^\mu\}$ is a basis of one-forms and an arbitrary one-form can be written as $\omega = \omega_\mu dx^\mu$. As we had obtained transformation law for vectors, for one forms,

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu$$

which gives, for components of one-forms

$$\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\omega_\mu \tag{3}$$