PHZ 6607: Special and General Relativity I
Lecture Notes

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Manifolds

In this lecture we discuss the concept of manifolds, and the more elementary notions of maps and charts. We start by looking at maps.

Basic Idea of a Map

One example of a map is a parametrized curve in some space, which is (local) map from that space to $\mathbb{R}$.

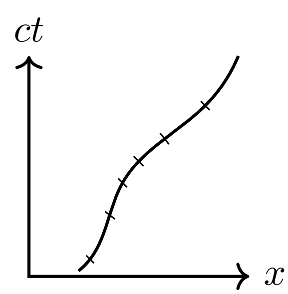

Figure 1: A parametrized curve in a 2-manifold x - ct

A map is a relationship between points in one space and points in another. In the above example the parametrized curve is a map from points in a two dimensional space-time to the real line. This map will be important when we introduce formally the concept of vectors in curved space-time.

Examples of 1-D Maps

Maps need not be continuous. An example of a discrete map is one that maps $1/n$ to n where n is a positive integer. The domain of this map is $(0, 1]$, while the image is $[1, \infty)$. It is an example of an invertible map, but it is not differentiable. On the other hand, the map of reals $x \in (0, 1)$ to $1/x \in (1, \infty)$ is both invertible and differentiable. Here we have excluded the end points, since we will be usually interested in maps from open intervals to open intervals. Note also that here we are mapping a finite domain to an infinite range.

Examples of 2-D Maps and Charts

We live in a four dimensional space-time so we will need to deal with more complicated maps than 1-D maps. Let's look at 2-D maps. We can map regions of a two dimensional space to $\mathbb{R}^2$ as shown in Figure 2, where we have three distinct maps. The images of these maps live in entirely different $\mathbb{R}^2$ spaces, and if want to relate them we would have to make use the inverses of these maps, assuming that they exist. (For example to go from the x^i space to the z^i space we would have to use the transformation $(\zeta \circ \phi^{-1})$). These regions might overlap or not. As a side note, are examples of topological spaces where there is non-zero overlap between every choice of neighborhoods of two distinct points. The spaces where every pair of points have disjoint neighborhoods are called **Hausdorff spaces**. Non-Hausdorff space are mathematical entities that play little to no role in physics (for more information see Wald and Hawking&Ellis).

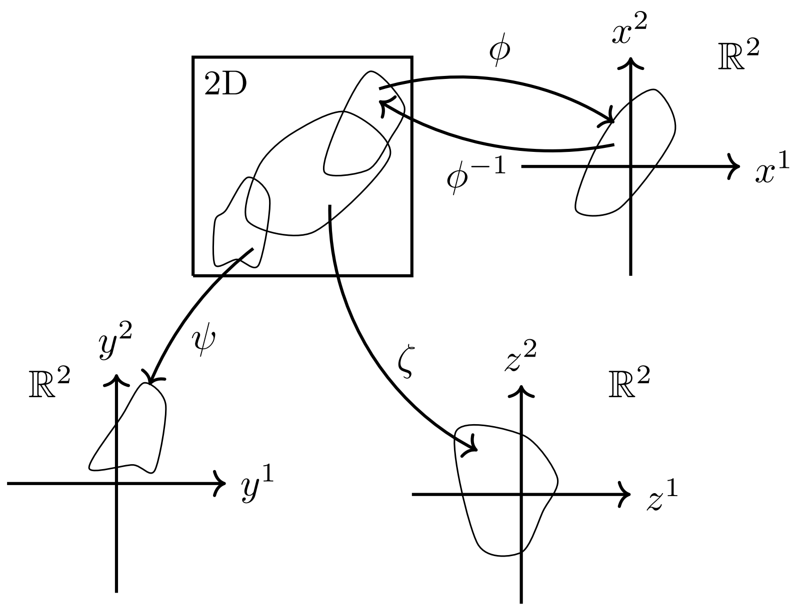

Figure 2: Two dimensional charts. A **chart** here consists of a subspace of a two dimensional space, along with a map from this subspace to $\mathbb{R}^2$.

We can also take $\mathbb{R}^2$ and map it into $\mathbb{R}^2$, by identifying a transformation of the coordinates. In this case we find that the whole of $\mathbb{R}^2$ can be mapped to $\mathbb{R}^2$ by a single map (for example by the identity map, see Figure 3).

The need for a collection of charts

However most of the maps we will encounter map only a subspace of the whole space into $\mathbb{R}^n$. The simplest example of that is the map from the circle the the real line (see Figure 4). Mapping points of the circle to the interval $(0, 2\pi)$ we see that the point of the circle that corresponds to 0 (or equivalently to 2π) is left out of this mapping, because we want our domain to be an open interval. Similarly mapping the circle to the interval $(\pi, 3\pi)$ we exclude the point of the circle at π. However, combining the domains of these two maps we can cover the whole circle and such a collection of charts is what we call an **atlas**. The maximum collection of all charts is what we call a **manifold**.

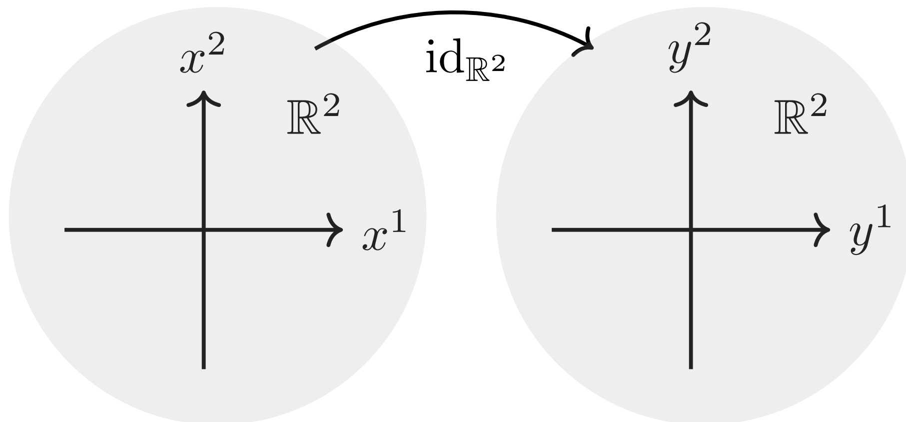

Figure 3: The whole of the Euclidean space $\mathbb{R}^2$ can be mapped to $\mathbb{R}^2$ with a single chart. Here we have chosen the identity map $\mathrm{id}_{\mathbb{R}^2}$, although obviously is choice is not unique.

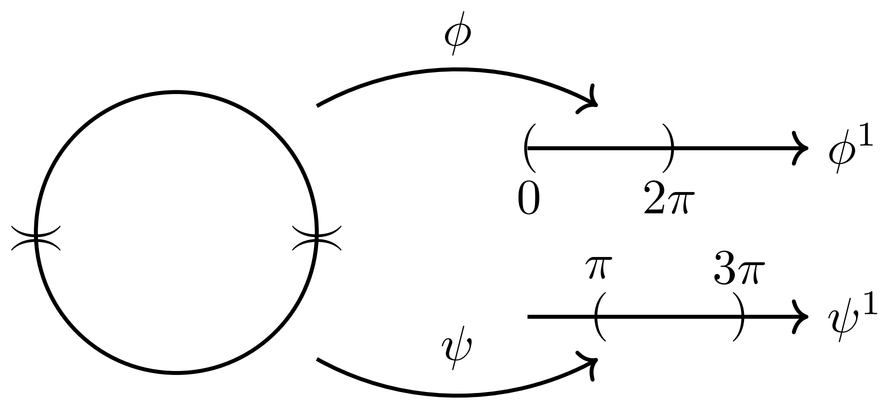

Figure 4: The collection of charts ϕ, ψ covers the whole circle, and therefore they constitute an atlas of the 1-sphere.

From Atlases to a Manifold

What do we mean by *maximum* collection of charts? Well, let's go back to Figure 2. The subregions drawn in the 2D domain, along with the mappings (ϕ, ψ, ζ), could in general be combined with other charts $(\xi, \rho, \ldots)$, until they covered the whole 2D space, and then the collection of charts $(\phi, \psi, \zeta, \xi, \rho, \ldots)$ would constitute an atlas of the original 2D space. We could call this Atlas 1. But obviously this would not be a unique choice of subregions of the 2D domain, nor a unique selection of mappings. Generally, we would be able to construct a different atlas, using charts whose images are also a subsets of $\mathbb{R}^2$, which we could call Atlas 2. We could even combine Atlas 1 and Atlas 2 to form a new atlas which would contain all the individual charts of atlases 1 and 2. Now the class of all possible atlases (that map to $\mathbb{R}^2$) is also an atlas, one which includes all possible (and compatible) charts of the 2D domain. In this sense, this atlas is maximal and a manifold is defined to be the set of all points in the 2D domain, along with this maximal atlas.

Atlases of a Sphere

We map every point every point of a sphere except the North Pole to a point on the plane (x, y) using the stereographic projection, as shown in Figure 5. Similarly, we can map all the points except the South pole to the plane (x, y) using again another stereographic projection. Together these two charts represent the manifold of the sphere. More common coordinates for the sphere are the spherical coordinates (θ, ϕ).

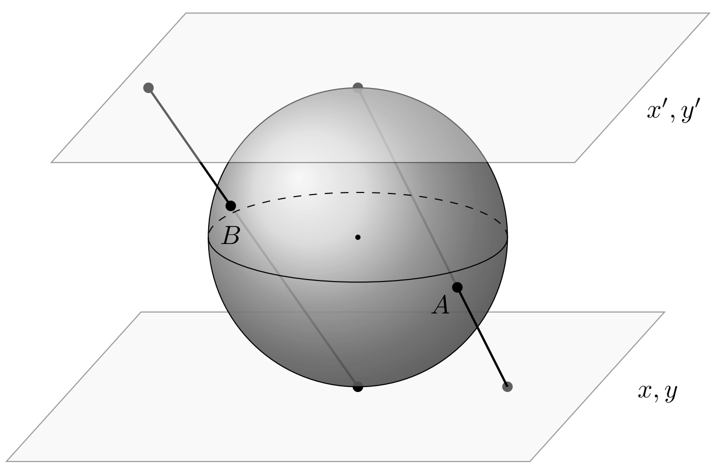

Figure 5: Each point of the sphere (except the North Pole) is mapped to the plane (x, y). Specifically the image of each point A on the sphere to the plane (x, y) is the intersection point between the plane (x, y) and the line defined by the North Pole and A. Similarly, the image of each point B on the sphere to the plane (x', y') is the intersection point between the plane (x', y') and the line defined by the South Pole and B.

Another chart of the sphere is that of coordinization using the angles θ and ϕ. This map breaks down all along the poles, since a single point of the sphere is mapped into a line of constant θ (as shown in Figure 6 and this makes the map non-invertible. Similarly, the curves (lines) at $\phi = 0$ and $\phi = 2\pi$ of Figure 6 correspond to

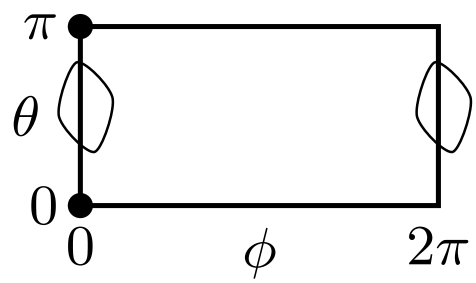

Figure 6: (θ, ϕ) chart of the sphere. The two small distinct patches in this space correspond to the same region on the sphere.

the same curve on the sphere. So this is an acceptable chart only in we exclude *from the sphere* both the poles and the line corresponding to $\phi = 0$. Note the the sphere is an example of a manifold without boundaries, while the space of Figure 6 *as a domain* would be considered a manifold with boundaries. For now we will restrict our discussion to manifolds without boundaries.

Examples and the Structure of Manifolds

From a physics point of view we need a local differentiable structure and rules to connect different charts whose domains overlap, which means that we want the mappings to be invertible. In Figure 7 we see different types of elementary one-dimensional maps. The Identity map, $y = x$ and $y = x^3$ are examples of differentiable and invertible maps. $y = x^2$ and $y = x^3 - x$ are differentiable, but not invertible mappings.

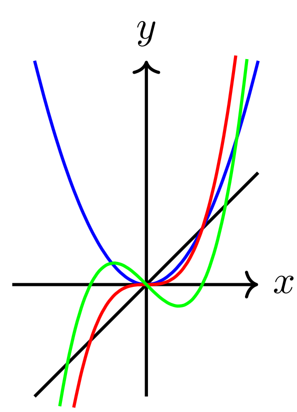

Figure 7: Black Curve: $y = x$, Blue Curve: $y = x^2$, Red Curve: $y = x^3$, Green Curve: $y = x^3 - x$

The map that rolls up the *whole* $\mathbb{R}^2$ plane to the cylinder (see Figure 8) is not invertible, but it is differentiable.

Moving on, we consider the two dimensional cone. One possible chart which would cover the whole cone

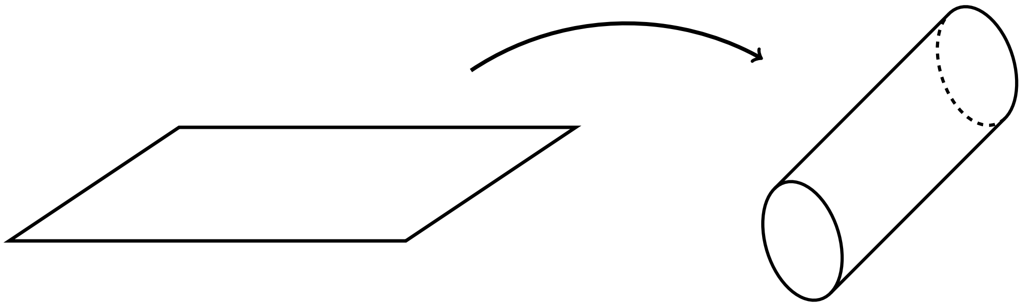

Figure 8: The mapping of an infinite plane rolled up to cylinder is not invertible.

would be that of the projection of the cone on the plane that defined by the basis of the cone (Figure 9). This map is invertible and differentiable everywhere except the apex of the cone.

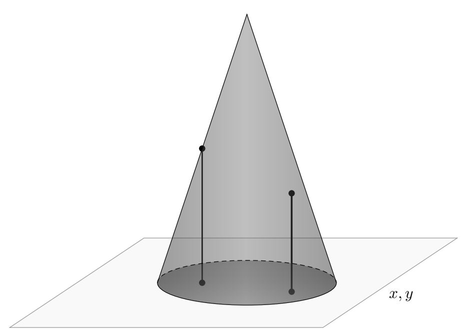

Figure 9: The projection of a cone to a plane is not differentiable at the apex of the cone.

Another map would be one where we introduce a cut on the surface of the cone and we unfold it into a two dimensional plane (Figure 10). From this chart, it is clear that the cone is flat [this property is not obvious in the projection chart because the projection changes the metric.]. On the other hand, the "cut" chart is not invertible at the red line of Figure 10, so we would need another chart, cut at some different line, in order to cover the cone.

Some examples of non-manifolds are the light cone, where the intersection point of the cones does not have a local differentiable structure and mathematical objects like a one-dimensional line attached to a two-dimensional plane.

As a side note, going back to the blue parabola of Figure 7, it would be an invertible map if we mapped the one-dimensional line x to the two-dimensional plane $(x, y = x^2)$ [instead of mapping it just to $y = x^2$]. Similarly, we could take the surface of the two-dimensional sphere and map it to the three-dimensional coordinates (x,y,z). But generally these maps whose image lives in a higher dimensional $\mathbb{R}^n$ will not be particularly useful to us.

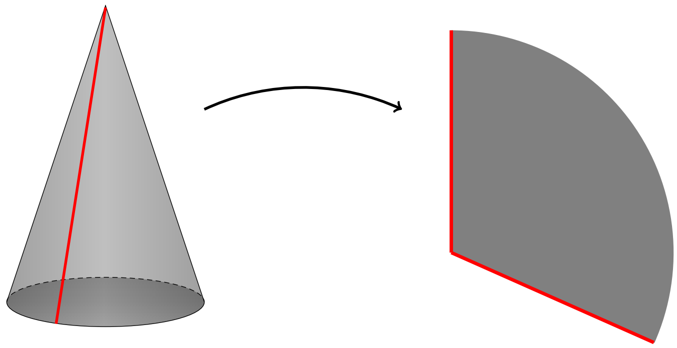

Figure 10: The cut chart is a map that unfolds the one-sided cone to a plane.

Additional Remarks about Manifolds

And as discussed above, all charts that are part of the same maximal atlas of a manifold will have the same dimensions for their image space. Throughout the course we will discuss charts that map a space $\mathbb{R}^n$ to a space $\mathbb{R}^m$, where n and m may or may not be the same. Different geometrical structures prevail in each of the three possible orderings of n and m.

Generally, intuition will be sufficient in our discussions of manifolds. Important exceptions to this rule are situations in Black Holes and cosmological spacetimes. For example, there are cosmological spacetimes where the past of one observer does not necessarily overlap with the past of another observer, with the two observers being space-like separated. The space that we live in is a good example of this situation which arises due to the expansion of the Universe. This poses of course restrictions to the part of the Universe that we can make predictions for. In particular, if we had a region of space that we know the properties of, we can propagate it to the future with certainty only inside the light-cone of this region [the cone with this region as its base].