Special and General Relativity PHZ 6607

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Geodesic equation derivation:

We start with the action we saw at the end of the last class.

$$S = \frac{1}{2}\int -g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}d\tau \tag{1}$$

We want to find the geodesic equation by using variational principles on the above given action.

$$\delta S = \frac{1}{2}\int -\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta g_{\mu\nu} + g_{\mu\nu}\frac{d\delta x^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{dx^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau} \tag{2}$$

$$\delta S = \frac{1}{2}\int -\frac{g_{\mu\nu}}{dx^\sigma}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\sigma + g_{\mu\nu}\frac{d\delta x^\mu}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau} \tag{3}$$

Integrating the second and the third terms by parts we get,

$$\delta S = \frac{1}{2}\int -\frac{g_{\mu\nu}}{dx^\sigma}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\sigma + \frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right)\delta x^\mu + \frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\right)\delta x^\nu \tag{4}$$

$$\delta S = \frac{1}{2}\int \left[g_{\sigma\nu}\frac{d^2x^\nu}{d\tau^2} + g_{\mu\sigma}\frac{d^2x^\mu}{d\tau^2} + \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right)\right]\delta x^\sigma d\tau \tag{5}$$

Since $\delta s = 0$, we have:

$$g_{\sigma\nu}\frac{d^2x^\nu}{d\tau^2} + g_{\mu\sigma}\frac{d^2x^\mu}{d\tau^2} + \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}) = 0 \tag{6}$$

And:

$$g^{\rho\sigma}\left[g_{\sigma\gamma}\frac{d^2x^\gamma}{d\tau^2} + g_{\mu\sigma}\frac{d^2x^\mu}{d\tau^2} + \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right)\right] = 0 \tag{7}$$

After contraction:

$$\frac{d^2x^\rho}{d\tau^2} + \frac{1}{2}g^{\rho\sigma}(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{8}$$

At this point we can define the christoffel symbols:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right) \tag{9}$$

Substituting which above, the geodesic equation will become:

$$\frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{10}$$

which can also be written as,

$$\frac{dx^\nu}{d\tau}\left[\frac{\partial}{\partial x^\nu}\frac{dx^\rho}{d\tau}+\Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tau}\right]=0 \tag{11}$$

The geodesic equation is an equation of motion and hence a general solution which is dependent on the metric. Depending upon how the metric behaves, we will get the appropriate solutions. Here we also make an important observation about the tensor nature of the Christoffel coefficients. The christoffel coefficients are composed of partial derivatives and hence **do not** transform as tensors of (1,2) type would transform. However, the bracket as a whole does transform as a tensor. If we notice carefully, there are no partial derivatives if the bracket is expanded fully and the partial derivatives are written as kronecker delta function. Hence, we redefine the bracket as,

$$Y^\rho_\nu=\frac{\partial}{\partial x^\nu}\frac{dx^\rho}{d\tau}+\Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tau} \tag{12}$$

and this newly defined quantity Y^ρ_ν **does** transform as a (1,1) tensor. Also, the first term $\frac{dx^\nu}{d\tau}$ is the velocity and will transform as a (1,0) tensor. This changes our geodesic equation to be written in a more simple form as,

$$A^\rho=V^\nu Y^\rho_\nu=0 \tag{13}$$

This action for which we have written the geodesic equation is not invariant under reparametrization as the standard action would be. For $S=\frac{1}{2}\int\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau$, this action would actually be invariant under reparametrization. A tranformation of the form $\tau\to\tau'=A\tau+B$, this action will only carry a constant with it and will end up giving the same geodesic equaiton, making it invariant under reparametrization.

1 Solutions to the geodesic equation:

Now we can look at the solutions of the geodesic equation starting from the simplest case of the flat space metric.

1.1 Flat Space Metric

If the spacetime is flat, $g_{\mu\nu}=diag(-1,1,1,1)$
Then we have $\frac{\partial g_{\sigma\nu}}{\partial x^\mu}=0$ and $\frac{d^2x^\rho}{d\tau^2}=0$ thus it gives us the solution is

$$x^\rho=x^\rho_0+v^\rho\tau \tag{14}$$

where τ is a parameter. For the reparmetrization invariant action, we typically normalize with the condition, $V^\mu V^\nu g_{\mu\nu}=-c^2$
to get the normalized action as $\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}=c$ which is the case for a timelike event. We can have different normalisation conditions which will give us $\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}=0$ which gives us the condition for the null motion or the light path.

Coming back to the flat space time, the action exactly will be:

$$s = -\frac{1}{2}\int\left[c^2\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2 - \left(\frac{dy}{d\tau}\right)^2 - \left(\frac{dz}{d\tau}\right)^2\right]d\tau \tag{15}$$

The canonical momentums are:

$$P_\mu = \frac{\delta s}{\delta(\frac{dx^\mu}{d\tau})} \tag{16}$$

Thus the canonical momentum corresponding to t is:

$$P_t = c^2\frac{dt}{d\tau} \tag{17}$$

Because the lagrangian doesn't depend on t or x^i for that matter, P_μ will also be a constant. For the particular case of P_t being a constant, the general solution will be $\tau = At + B$, where A and B are some constants. Similarly we can find that P_x, P_y, P_z are also constant.

The same metric in spherical coordinates will be written as,

$$s = -\frac{1}{2}\int\left[c^2\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]d\tau \tag{18}$$

Now, in spherical coordinates P_t, P_ϕ are still constant, but P_r, P_θ are no longer constant. However, L_x and L_y are still constant along with the other components which is consistent with the previous result.
If we make a Lorentz transform, the action in the new system is,

$$s = -\frac{1}{2}\int\left[c^2\left(\frac{dt'}{d\tau}\right)^2 - \left(\frac{dx'}{d\tau}\right)^2 - \left(\frac{dy'}{d\tau}\right)^2 - \left(\frac{dz'}{d\tau}\right)^2\right]d\tau \tag{19}$$

$$\begin{aligned} x' &= \gamma(x - ut) \\ t' &= \gamma(t - ux/c^2) \end{aligned}$$

Where u is the relative velocity between two systems, and $\gamma = 1/\sqrt{1 - u^2/c^2}$. We can also find the reverse transform:

$$\begin{aligned} x &= \gamma(x' + ut') \\ t &= \gamma(t' + ux'/c^2) \end{aligned} \tag{20}$$

If $x = ut$, $t = \gamma t'$ and $x' = 0$, and t' is called the proper time. In this arrangement we can now compare two situations, one where a particle travels the $x = vt$ path reaches a point(the green path) and another when we travel along the light path(see diagram) $x = ct$ followed by $x = -ct$ which takes us to the same position.