Special and General Relativity
PHZ6607 - Fall/2018

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1) SPACE

Consider a two-dimensional Euclidean space, with two different coordinate systems, in which one is the rotation of the other by an angle ϕ:

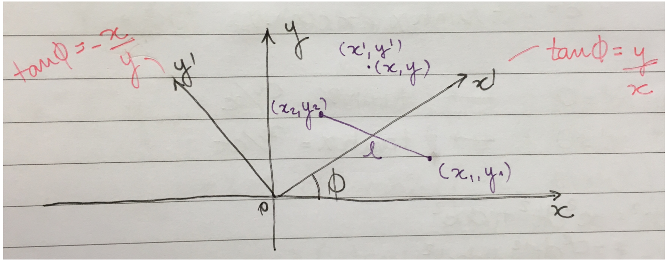

Figure 1

For the rotation in Figure 1, the transformed coordinates x' and y' will be given by

$$
\begin{aligned}
x' &= x\cos\phi + y\sin\phi \\
y' &= -x\sin\phi + y\cos\phi
\end{aligned}
\tag{1}
$$

The distance between two points l^2 is invariant under this transformation. Thus

$$
l^2 = \Delta x^2 + \Delta y^2 = \Delta x'^2 + \Delta y'^2, \tag{2}
$$

and so is the infinitesimal interval (or line element) ds^2:

$$
ds^2 = dx^2 + dy' = dx'^2 + dy'^2. \tag{3}
$$

2) SPACE-TIME

Now consider a space-time diagram, in which the $\{ct', x'\}$ coordinates are representing a boost in the $\{ct, x\}$ coordinates:

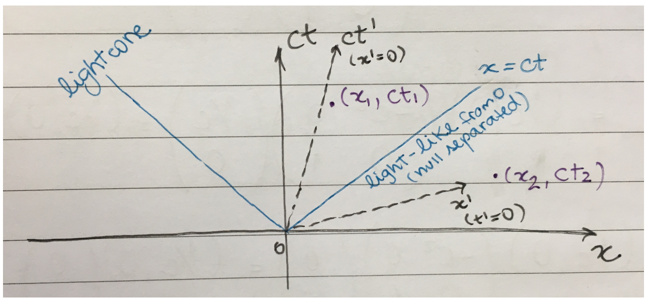

Figure 2

Note that the light cone does not change, and that the space and time axes scissor together instead of remaining orthogonal in the traditional Euclidean sense (although they do in the Lorentzian sense). The line element in this case is

$$ds^2 = -c^2dt^2 + dx^2. \tag{4}$$

The distance l_1^2 between the point (x_1,ct_1) - indicated in Figure 2 - and the origin is

$$l_1^2 = -c^2t_1^2 + x_1^2 < 0, \tag{5}$$

which represents a time-like separation from the origin. And in the case of the point (ct_2,x_2), this distance is expressed by l_2^2:

$$l_2^2 = -c^2t_2^2 + x_2^2 > 0, \tag{6}$$

representing a space-like separation from the origin. So the distance between those two points is

$$\Delta l^2 = -c^2(t_2 - t_1)^2 + (x_2 - x_1)^2, \tag{7}$$

which can be positive or negative.

Considering that

$$\begin{aligned} ct' &= ct\cosh(\chi) - \chi\sinh(\chi) \\ x' &= -ct\sinh(\chi) + x\cosh(\chi), \end{aligned} \tag{8}$$

if $t' = 0 \rightarrow \tanh(\chi) = ct/x$;

if $x' = 0 \rightarrow \tanh(\chi) = x/ct$;

if $x' = 0$ and $\tanh(\chi) = v/c \rightarrow x = vt$.

3) ORTHOGONALITY

Let's begin by making $\hat{X} = (0, 1)$ and $\hat{T} = (1/c, 0)$. Thus, we have:

$$\hat{X}^\mu \hat{X}^\nu g_{\mu\nu} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -c^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \tag{9}$$

$$\hat{T}^\mu \hat{T}^\nu g_{\mu\nu} = \begin{pmatrix} 1/c & 0 \end{pmatrix} \begin{pmatrix} -c^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/c \\ 0 \end{pmatrix} = \begin{pmatrix} 1/c & 0 \end{pmatrix} \begin{pmatrix} -c \\ 0 \end{pmatrix} = -1 \tag{10}$$

$$\hat{X}^\mu \hat{T}^\nu g_{\mu\nu} = \begin{pmatrix} 1/c & 0 \end{pmatrix} \begin{pmatrix} -c^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \tag{11}$$

which means that $\hat{X}$ and $\hat{T}$ are orthogonal. However, if we consider $\vec{V} = (1/c, 1)$, then $V^\mu V^\nu g_{\mu\nu} = 0$, because it's time-like. And in the case of $\vec{U} = (-1/c, 1)$, $U^\mu U^\nu g_{\mu\nu} \neq 0$.

– ***Note***: below onward, we will use (ct,x), as opposed to earlier when we used (x,ct).

Consider now that $\vec{X} = (x\tanh(\chi), x)$ and $\vec{T} = (ct, ct\tanh(\chi))$. Thus

$$X^\mu T^\nu g_{\mu\nu} = \begin{pmatrix} ct & ct\tanh(\chi) \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x\tanh(\chi) \\ x \end{pmatrix} \tag{12}$$

$$X^\mu T^\nu g_{\mu\nu} = \begin{pmatrix} ct & ct\tanh(\chi) \end{pmatrix} \begin{pmatrix} -x\tanh(\chi) \\ x \end{pmatrix} \tag{13}$$

$$X^\mu T^\nu g_{\mu\nu} = xct(-\tanh(\chi) + \tanh(\chi)) = 0. \tag{14}$$

In this case, $\vec{X}$ and $\vec{T}$ are orthogonal, and $(0, x')$, $(ct', 0)$ are orthogonal in $g_{\mu'\nu'}$.

$$\begin{pmatrix} -x\tanh(\chi) & x \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -x\tanh(\chi) \\ x \end{pmatrix} = x^2(1 - \tanh^2(\chi)) \tag{15}$$

4) <u>COORDINATE TRANSFORMATIONS</u>

ROTATION

If we want to make a rotation of angle ϕ around the z-axis, we should use the matrix

$$R^z_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{16}$$

This can be extended to the other axes by interchanging the second line/column with the third line/column for a rotation around the y-axis (R^x_Ψ); and then the first line/column with the second line/column for a rotation around the x-axis (R^y_γ). $\{R^x_\Psi, R^y_\gamma, R^z_\phi\}$ represents a non-abelian rotation group.

BOOST

If we want to make a boost in the x-direction by an angle χ:

$$B^x_\chi = \begin{pmatrix} \cosh(\chi) & -\sinh(\chi) & 0 \\ -\sinh(\chi) & \cosh(\chi) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{17}$$

which can also be extended to the other axes to find B^y_η and B^z_ρ.

$\{B^x_\chi, B^y_\eta, B^z_\rho\} + \{R^x_\Psi, R^y_\gamma, R^z_\phi\}$ represents a larger non-abelian group.

By performing the boost, we have that:

$$ct' = ct\cosh(\chi) - x\sinh(\chi) = \cosh(\chi)(ct - x\tanh(\chi)) = c\gamma(t - vx/c^2) \tag{18}$$

$$x' = -ct\sinh(\chi) + x\cosh(\chi) = \gamma(x - vt) \tag{19}$$

5) <u>GEODESICS</u>

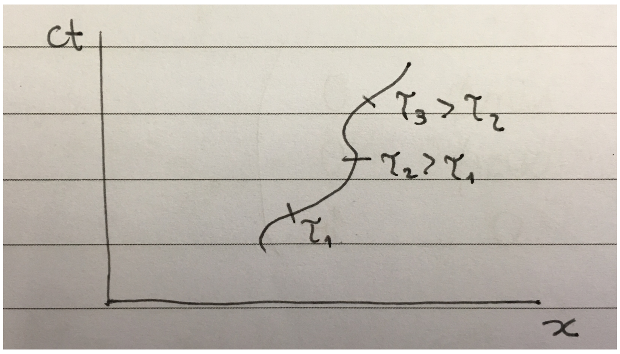

Figure 3

The extremal path corresponds to no forces (geodesics), and the proper time τ is reparametrization invariant. To find the length along the path S:

$$S = \int \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau \tag{20}$$

To be better discussed in the next class.