# Fundamentals of Vector and Tensors

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## 0.1 Types of Vectors

There are two types of vectors we will be working with. These are defined as covariant(1-Form, covector) and contra-variant (vector). Such as, a covector $W_i$ and a vector $V^j$.

### 0.1.1 Raising and Lowering Indicies

To see this, we raise and lower the indcies of a vector $V^j$ and a covector $W_i$ and a some tensor $g_{ij}$, but before that here are some tips: We want to take a lower index and raise it to a higher index. We will apply a tensor $g_{ij}$ or $g^{ij}$ to achieve this. We want to sum over the second index in the tensor.

$$V^i = g^{ij} V_j \tag{1}$$

$$W_i = g_{ij} W^j \tag{2}$$

Now the order of $gV$ or $Vg$ does not matter. What does matter, however, is which index you are summing over. For example, $V^i = g^{ij} V_j$ means you are summing over the second index of the metric tensor, j: $g^{ij} V_j = g^{i0} V_0 + ... g^{i3} V_3 = V^i$. It does not matter if the $V$ comes before or after the $g$ because $V_{0,1,2,3}$ is just a number. However, if I were to write $g^{ji} V_j = g^{0i} V_0 + ... g^{3i} V_3 = V^i$ (notice it is $g^{ji}$ and not $g^{ij}$), I would be summing over the first value j. Since we want to sum over only the second index, we discard these types of notations as wrong. In the specific case that $g_{ij}$ is the metric tensor, then because it is symmetric, $g_{ij} = g_{ji}$, we can permute our tensors however we like, and not have to worry about the above notational problem.

### 0.1.2 Transformation between Coordinates

Now suppose we want to ask the question, "How do my covetors and vectors look in a different basis?". The coordinate transformation for this is done in the following way:

$$V^{j'} = \frac{\partial x^{j'}}{\partial x^i} V^i \tag{3}$$

$$W_{j'} = \frac{\partial x^i}{\partial x^{j'}} W_i \tag{4}$$

Example: Suppose we have a curve $x^\alpha(\lambda)$ on some manifold $M$, parametrized by $\lambda$. The tangent vector at a point P is $u^\alpha = \frac{dx^\alpha}{d\lambda}|_P$. $u^\alpha$ is a vector and transforms like (3). Now suppose we have a scalar function $f(x^\alpha)$, then the gradient of this scalar function at a point P, $\partial_\alpha f = \frac{\partial f}{\partial x^\alpha}|_P$, is a covector and transforms like(4).

## 0.2 Tensors

We define a tensor of rank(n,m) = n+m, to be a tensor with n "up-stair" indicies and m "down-stair" indicies. For example, $g_{ij}$ has two bottom indicies, hence it is a rank(0,2) = 2 tensor. A tensor like, $g^i_j$ has rank(1,1) = 2. Using this we can understand what rank tensors are covectors and vectors. A vector like $V^i$ will have rank(1,0) = 1 and a covector like $W_j$ will have rank(0,1) = 1.

Now we ask a question, "How does the metric tensor function". We know it takes a set of vectors and covectors and spits out a real number at the end(i.e. is a map between covectors/vectors to a real number). The mechanical point of this is that it uses contraction to calculate a scalar which is a real number. That is, $g_{ij}(V^i, V^j) \to |V|^2 \to R$. The magnitude need not be positive. A similar description can be done for covectors except in this case one should use $g^{ij}$ instead for proper contraction. The physical point to hit home is that a contraction returns a scalar. GR says it does not matter what coordinate system you choose. Scalars are coordiante invariant quantities. Hence, they are physically observable/measurable, making it physics.

*When we write primes on the index we are calcualting something across different tangent spaces. When we don't write primes, we are raising or lower an index within a single tangent space.

### 0.2.1 Transformation between Coordiantes

We think of tensors, like all vector fields evaluated at some point P on a manifold M, as lying in the tangent plane to the point P. We can add, subtract, and contract tensors within the same tangent space, however we cannot as easily relate one tensor on one tangent plane at point P to another tensor on different tangenet plane at some point Q. So we need to write down a method to relate tensors from different tangent spaces. We will first write out the transformation and then understand how it occured.

$$T^{\alpha' \ldots \beta'}{}_{\gamma' \ldots \delta'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \cdots \frac{\partial x^{\beta'}}{\partial x^\beta} \frac{\partial x^\gamma}{\partial x^{\gamma'}} \cdots \frac{\partial x^\delta}{\partial x^{\delta'}} T^{\alpha \ldots \beta}{}_{\gamma \ldots \delta} \tag{5}$$

One can see, referring to equations (3) and (4), that the tensor transforms like a vector for the upper indicies and a covector for the lower indicies.

## 0.3 Final Remarks

### 0.3.1 On Vectors

Suppose I write $V = V^i \vec{e_i}$. A priori, $V$ can be defined in any basis I choose, so what the above statment actually means is I chose to write the vector $V$ in the following basis $\vec{e_i}$ with the following components $V^i$. Now we can take the dot product which is calculated by a contraction:

$$V \cdot V = V^i \vec{e_i} \cdot V_j \vec{e^j} = V^i V_j \left( \vec{e_i} \cdot \vec{e^j} \right) = V^i V_j \delta_i^j = |V|^2 \tag{6}$$

Notice we could have only written the dot product of the basis vectors as a kronecker delta, because we chose to work with independent basis vectors of the same tangent space.

*Basis vectors like $\vec{e_\alpha}$ transform like covectors. *Components like $V^i$ transform like vectors.

### 0.3.2 On Tensors

Now, not every two index object is a tensor. For example the coordinate transformation of a covector and vector, respectively, is:

$$V_{\mu'} = \frac{\partial x^\nu}{\partial x^{\mu\prime}} V_\nu = V_\nu \Lambda^\nu_{\mu'}. \tag{7}$$

$$V^{\mu'} = \frac{\partial x^{\mu\prime}}{\partial x^\nu} V^\nu = V^\nu \Lambda^{\mu'}_\nu. \tag{8}$$

Although $\Lambda$ has two indicies, it is not a tensor because it is defined in two different bases, one of the primes coordinates and one of the unprimes coordiantes. Recall, tensors exist in tangent spaces so they are specific to a single tangent space. Finally I can write the following:

$$V^i W_j V^k W_l W_p V^q = T^i{}_j{}^k{}_{lp}{}^q \tag{9}$$

In order for us to be careful with the notation, the following is generally NOT equivalent:

$$T^i{}_j{}^k{}_{lp}{}^q \neq T^{ikq}{}_{jlp} \tag{10}$$

The index order matters i.e. from left to right. But we can always raise or lower indicies via metric. If we contract a tensor of rank(3,3) = 6 the final tesnor will have one less index upstairs and one less downstairs, so rank(3-1,3-1) = (2,2)=4: $T^i{}_j{}^k{}_{lp}{}^q \rightarrow T^i{}_j{}^k{}_{ip}{}^q \rightarrow U_j{}^k{}_p{}^q$ .