11/30/2018 Notes

Grant Elliot

1. Perturbations

Assuming $h_{\mu\nu} << 1$ we can write the physical metric as

$$g^{phys}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \tag{1}$$

$g_{\mu\nu}$ is the background metric and $h_{\mu\nu}$ is a perturbation.

The Einstein tensor of the background metric will typically be either $T_{\mu\nu}$, $-\Lambda g_{\mu\nu}$ or 0. The Einstein tensor of the physical metric will be

$$G(g^{phys}_{\mu\nu}) = G(g_{\mu\nu}) + F(h_{\mu\nu}) = G(g_{\mu\nu}) - \frac{1}{2}E(h_{\mu\nu}) \tag{2}$$

E is some second order operator. Using the equations for the Christoffel symbols and the variation of the metric

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2}g^{\mu\tau}\Big(\frac{\partial g_{\tau\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\tau}}{\partial x^{\sigma}} - \frac{\partial g_{\nu\sigma}}{\partial x^{\tau}}\Big) \tag{3}$$

$$\delta g^{\mu\tau} = -g^{\mu\rho}\delta h_{\rho\alpha}g^{\alpha\tau} \tag{4}$$

$$\delta\Gamma = -g^{\mu\rho}\delta h_{\rho\alpha}\Gamma^{\alpha}_{\nu\sigma} + \frac{1}{2}\Big(\frac{\partial h_{\tau\sigma}}{\partial x^{\nu}} + \frac{\partial h_{\nu\tau}}{\partial x^{\sigma}} - \frac{\partial h_{\nu\sigma}}{\partial x^{\tau}}\Big) \tag{5}$$

$$= \frac{1}{2}g^{\mu\tau}\Big(\nabla_{\nu}h_{\tau\sigma} + 2\Gamma + \nabla_{\sigma}h_{\nu\tau} + 2\Gamma - \nabla_{\tau}h_{\nu\sigma} - 2\Gamma\Big) \tag{6}$$

This results in

$$\delta\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2}g^{\mu}\tau(\nabla_{\nu}h_{\tau\sigma} + \nabla_{\sigma}h_{\nu\tau} - \nabla_{\tau}h_{\nu\sigma}) \tag{7}$$

The variation of the Ricci tensor give $\partial\Gamma + \Gamma\delta\Gamma$, and perturbations will give $\partial\delta\Gamma + \Gamma\delta\Gamma$. These conspire to make $\nabla\delta\Gamma$. This will make E end up to be

$$E_{ab} = \nabla^{c}\nabla_{c}h_{ab} + \nabla_{a}\nabla_{b}h^{c}_{c} - \nabla^{c}\nabla_{(a}h_{a),b} + g_{ab}(\nabla^{c}\nabla^{d}h_{cd} - \nabla^{c}\nabla_{c}h^{d}_{d}) \tag{8}$$

For a small test particle

$$E_{ab} = -16\pi T_{ab}(\epsilon) \tag{9}$$

Under a gauge transformation

$$g \to g' = g + \mathcal{L}_{\xi}g \tag{10}$$

we have

$$h \to h' = h + \mathcal{L}_{\epsilon\xi}g \tag{11}$$

It took 30 years for the initial value formulation of general relativity to be developed.

2. Lorenz Gauge

The Lorenz gauge is written in terms of

$$\tilde{h}_{ab} = h_{ab} - \frac{1}{2} g_{ab} h^c_c \tag{12}$$

The trace is now reversed

$$\tilde{h}^c_c = h^c_c - \frac{1}{2} 4 h^c_c = -h^c_c \tag{13}$$

The constraint is $\nabla^a \tilde{h}_{ab} = 0$. E takes the form

$$E = \nabla_c \nabla_c \tilde{h}_{ab} + 2 R_{a\ b}^{\ c\ d} \tilde{h}_{cd} = -16\pi T_{ab} \tag{14}$$

This is still not easy to solve as it is ten coupled linear pdes.

In flat space the Riemann curvature tensor is zero the the equation reduces to

$$\nabla^c \nabla_c \tilde{h}_{ab} = 0 \tag{15}$$

Write e^{ikx} where $kx = k \cdot x$, k and x are four vectors and $k = \omega(1, n^x, n^y, n^z)$. Then h_{ab} can be expressed as

$$\tilde{h}_{ab} = C_{ab} e^{ikx} \tag{16}$$

We also have $k^a C_{ab} = 0$ which reduces the number of independent components to 6. Using additional gauge freedom with $\nabla^c \nabla_c \xi = 0$

$$\tilde{h} \to \tilde{h}' = \tilde{h} + \mathcal{L}_\xi g \tag{17}$$

Using gauge freedom to remove additional terms results in

$$\begin{aligned} \tilde{C}^a{}_a &= 0 \\ \tilde{C}_{0a} &= 0 \end{aligned} \tag{18}$$

There are two degrees left, this is transverse traceless gauge. There are plane propagating gravitational waves in perturbed Minkowski space.

The metric can be split into the components, $g_{tt}, g_{ti}, g_{it}, g_{ij}$. Under the rotation subgroup g_{tt} will behave as a scalar, g_{tj} will transform as a covector, and g_{ij} will behave like a second rank tensor. Given foliated spacetime one has the freedom to rotate on every Cauchy slice.

The metric can also be split into (t, r) coordinates as well as (θ, ϕ) coordinates. If a, b denote t, r and i, j denote θ, ϕ coordinates, the h_{ab} is a scalar. h_{ij} is a spin two tensor. There are also two vectors which can be written in terms of there transverse and longitudinal components. The divergence of the transverse part is zero while the divergence of the longitudinal component is a scalar. There is then one degree of freedom in the transverse component.