Special and General Relativity PHZ 6607

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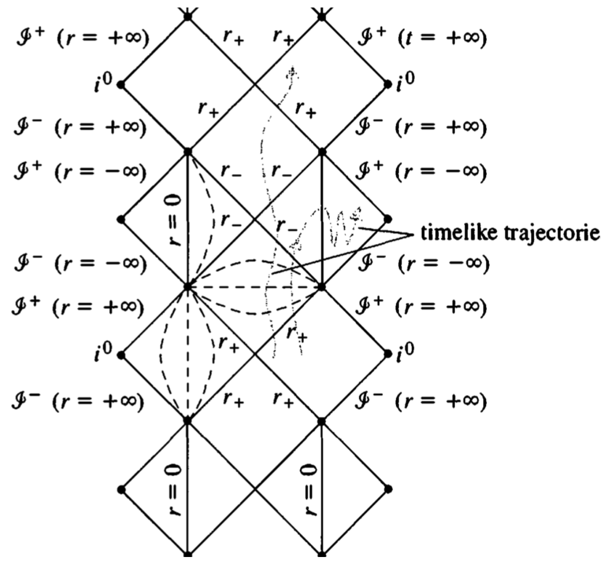
1 Kerr BH Conformal Diagram

We start with the Kerr Metric,

$$ds^2 = \left(1 - \frac{2GMr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{sin^2\theta}{\rho^2} \Sigma^2 d\phi^2 - \frac{2GMarsin^2\theta}{\rho^2}(dtd\phi + d\phi dt) \tag{1}$$

$$\rho^2 = a^2 cos^2\theta + r^2 \qquad \Delta = r^2 - 2GMr + a^2 \qquad \Sigma^2 = [(r^2 + a^2)^2 - a^2 \Delta sin^2\theta]$$

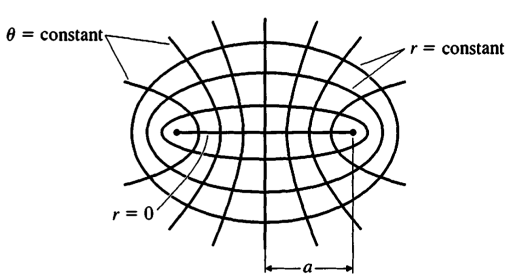
and the conformal diagram looks like,

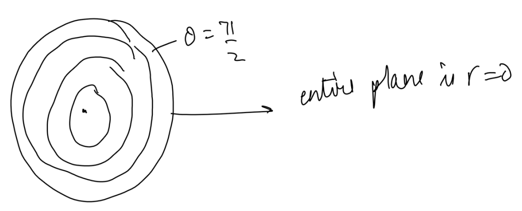


Here we must note that each point on the conformal diagram is a closed surface and hence the geometries are different at each of these points. The event horizons in this geometry are the zeros of Δ and are given by, $r_\pm = M \pm \sqrt{M^2 - a^2}$. Here we keep in mind that these are event horizons and not the Killing Horizons. Just looking at the simple expressions, we can see that the there are three cases,

$$M > a > 0 \implies r_\pm > 0, r_\pm \in Real$$
$$a < 0 \implies NakedSingularity$$
$$a = M \implies Unstablesolution$$

We are only interested in the case where we see a real and positive solution for the event horizons.

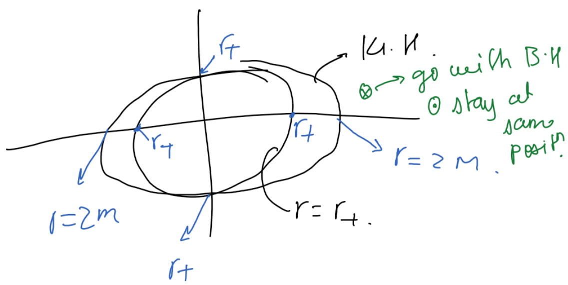




In the first figure we see a r, θ representation of a point in the conformal diagram. The hyperbolic lines are all $\theta =$ constant lines and the elliptical lines are $r =$ constant lines in the geometry. The dots we see highlighted in the first image can be drawn elaborately in the second image. Hence, the second image is a mere projection of the first image.

Another point to note here are the Killing Vectors in this metric. As none of the components in the metric ar a function of ϕ or t, we have $T = \partial_t$ and $\Phi = \partial_t$ as our two first Killing Vectors.

If we look at the side projection of the conformal diagram,



We can see that the Killing Horizons are at $r = 2M$ at $\theta = \pi/2$, whereas the event horizons are at $r = r_{\pm} \forall \theta$. In this geometry the Killing Vector, T, which is timelike outside the event horizon changes and becomes spacelike as it goes inside the event horizon. In the image above, a particle travelling with the BH along it's direction of rotation, the particle will rotate. However, a particle in the opposing direction of the spin will stay at the same position.

2 Angular Momentum

We consider null geodesics in the equitorial plane to make our life simple. So as a typical stable equitorial orbit,

$$\dot{r} = 0 \qquad \dot{\theta} = 0 \qquad \theta = \frac{\pi}{2}$$

This leaves us with,

$$0 = ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \frac{4GMa}{r} dt d\phi + \frac{[(r^2 + a^2)^2 - a^2]}{r^2} d\phi^2$$

$$\frac{\partial \phi}{\partial t} = \frac{g_{t\phi} + \sqrt{g_{t\phi}^2 - g_{tt} g_{\phi\phi}}}{g_{\phi\phi}}$$

And in the stationary limit surface of the Kerr Metric,

$$g_{tt} = 0$$

And the two solutions are,

$$\left.\frac{d\phi}{dt}\right|_{g_{tt}=0} = 0 \qquad \left.\frac{d\phi}{dt}\right|_{g_{tt}=0} = \frac{2g_{t\phi}}{g_{\phi\phi}}$$

$$\left.\frac{d\phi}{dt}\right|_{g_{tt}=0} = 0 \qquad \left.\frac{d\phi}{dt}\right|_{g_{tt}=0} = \frac{a}{2M^2 + a^2}$$

and we can also find,

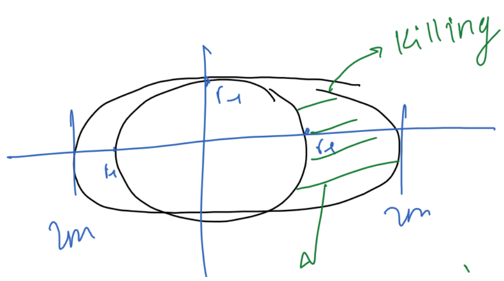
$$\left.\frac{d\phi}{dt}\right|_{r=r_+} = \frac{2Ma}{4M^2(M+\sqrt{M^2-a^2})} \qquad\qquad = \frac{a}{a^2+r_+^2} = \Omega_H$$

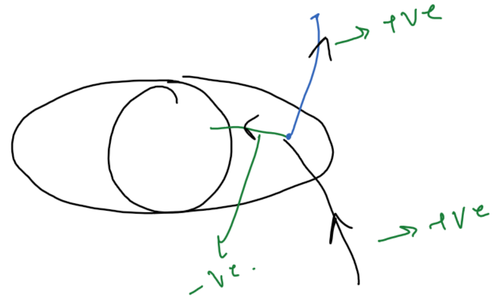
The Killing Vector associated with this is,

$$K^\mu = T^\mu + \Omega_H \Phi^\mu$$

and this makes $r_\pm$ a Killing Horizon for this Killing Vector.

3 Penrose Process





In the ergo region, we have $E = -T_\mu P^\mu < 0$ as T_μ is spacelike. The plunge however occurs when the $E > 0$. If you are falling in the horizon, your energy is already +ve, If you throw a particle inside the $r = r_+$ boundary, $E_{particle} < 0$

$\implies$ energy after > energy before $\implies$ BH energy goes down $\implies$ M_{BH} goes down. The reduction in this mass can be done repetitively until the all the rotational energy of the Black Hole is over. After that point the energy or the mass won't go down. At this point, what remains

$$\begin{aligned} M_{in}^2 &= \frac{A_+}{4\pi} \\ &= \frac{r_+^2 + a^2}{4\pi} \\ &= \frac{2Mr_+}{4\pi} \end{aligned}$$

where the amount of mass lost is $\Delta M = M - M_{ir} \leq M\left(1 - \frac{1}{\sqrt{2}}\right)$