Special & General Relativity I

Suzanne Rosenzweig & Corey Bathurst

November 7, 2018

1 Static Schwarzschild Star

In order to understand gravitational collapse to a black hole, we should first understand static configurations describing the interiors of spherically symmetric stars. Consider the general static, spherically symmetric metric in a static vacuum

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega \tag{1}$$

for which the Christoffel symbols, the Riemann tensors, and the Ricci tensors have been calculated in the beginning of chapter 5. From these, the curvature scalar can be given by

$$R = -2e^{-2\beta}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}(\partial_r\alpha - \partial_r\beta) + \frac{1}{r^2}(1 - e^{2\beta})\right] \tag{2}$$

The Einstein tensor can be calculated using the Ricci tensor and curvature scalar and is given by

$$G^t_t = -\frac{1}{r^2}e^{-2\beta}\left(2r\partial_r\beta - 1 + e^{2\beta}\right) \tag{3a}$$

$$G^r_r = \frac{1}{r^2}e^{-2\beta}\left(2r\partial_r\alpha + 1 - e^{2\beta}\right) \tag{3b}$$

$$G^\theta_\theta = e^{-2\beta}\left(\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{1}{r}(\partial_r\alpha - \partial_r\beta)\right) \tag{3c}$$

$$G^\phi_\phi = \frac{1}{\sin^2\theta}G^\theta_\theta. \tag{3d}$$

Initially considering the absence of a source, Eqns. 3a and 3b go to zero independently, meaning we can set their difference equal to zero, which allows us to solve for α in terms of β.

$$\begin{aligned} 2r\partial_r\beta - 1 + e^{2\beta} + 2r\partial_r\alpha + 1 - e^{2\beta} &= 0 \\ 2r\partial_r\beta + 2r\partial_r\alpha &= 0 \\ \alpha &= -\beta \end{aligned} \tag{4}$$

By setting Eqn. 3a equal to zero, it is possible to solve for $\beta(r)$. Make the substitution $V = e^{-2\beta}$ and use $V' = -2V\partial_r\beta$ to compute

$$\begin{aligned} 2r\partial_r\beta - 1 + e^{2\beta} &= 0 \\ r\frac{V'}{V} - 1 + \frac{1}{V} &= 0 \\ rV' - V + 1 &= 0 \\ -(rV)' + 1 &= 0 \\ [r(V-1)]' &= 0. \end{aligned} \tag{5}$$

Since this is a first derivative equal to zero, we can set the function inside the brackets equal to an integration constant with the same dimensions as r (since V is dimensionless).

$$r(V-1) = \text{constant} \equiv -r_S \tag{6}$$

The integration constant r_S is interpreted as the Schwarzschild radius and we must define it in terms of some physical parameter. Since the Schwarzschild metric should reduce to the weak-field case when $r \gg GM$

$$g_{tt} = -\left(1 - \frac{2GM}{rc^2}\right) \tag{7}$$

then we must identify

$$r_S = \frac{2GM}{c^2} \tag{8}$$

which gives us

$$V = 1 - \frac{2GM}{rc^2}. \tag{9}$$

Since we found that $\alpha = -\beta$, we have the relation

$$e^{2\alpha} = e^{-2\beta} = V = 1 - \frac{2GM}{rc^2}. \tag{10}$$

To proceed, we will have to use the energy-momentum tensor and the conservation equations

$$\nabla_\mu G^\mu_\nu = 0 \rightarrow \nabla_\mu T^\mu_\nu = 0. \tag{11}$$

If we add a constant to the Einstein tensor, which means changing the Einstein equations from

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \rightarrow G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{12}$$

where Λ is a cosmological constant, it's obvious that the covariant divergence of this side will also be zero if lambda is a constant so this will still be conserved. It should be noted that Λ is not a constant of integration (like r_S) and that it could have come from the Lagrangian.

When we discuss the rotating black hole, we will see that there is no real way you can write down the solution in terms of integrations in this way. One must use Cauchy-Riemann's equations and write down a hypothesis for the form of solution, which has constants in it; the solution holds for any value of those constants but you can never get their values by simple integration. Unlike a stationary black hole, which is defined only by its mass, a rotating black hole is defined by two parameters: mass and angular momentum. This is why Schwarzschild was able to solve Einstein's equations mere months after Einstein published his theory but then it took almost 50 years for someone to find a solution which corresponds to a vacuum spacetime with rotation.

To proceed, we model the star itself as a perfect fluid, with energy-momentum tensor in the rest frame given by

$$T_{\mu\nu} = \begin{pmatrix} -\rho(r) & & & \\ & p(r) & & \\ & & p(r) & \\ & & & p(r) \end{pmatrix} \tag{13}$$

For a perfect fluid, we require $\rho(p)$ in order to solve the conservation equation. The only conservation equation that matters is when $\nu = r$ because every other value of ν will give only zero.

$$\nabla_\mu T^\mu_r = \partial_r T^r_r + \Gamma^\mu_{\lambda\mu} T^\lambda_r - \Gamma^\lambda_{r\mu} T^\mu_\lambda = 0. \tag{14}$$

The only Christoffel symbols that contribute here are

$$\Gamma^t_{tr} = \partial_r \alpha \tag{15a}$$
$$\Gamma^r_{rr} = \partial_r \beta \tag{15b}$$
$$\Gamma^\theta_{r\theta} = \frac{1}{r} \tag{15c}$$
$$\Gamma^\phi_{r\phi} = \frac{1}{r} \tag{15d}$$

and the derivative is only nonzero if $\lambda = r$ so only one derivative survives and due to the fact that the energy-momentum tensor is diagonal, the conservation equation becomes

$$\begin{aligned}
\nabla_\mu T^\mu_\nu &= \partial_r T^r_r + \Gamma^t_{rt} T^r_r + \Gamma^r_{rr} T^r_r + \Gamma^\theta_{r\theta} T^r_r + \Gamma^\phi_{r\phi} T^r_r - \Gamma^t_{rt} T^t_t - \Gamma^r_{rr} T^r_r - \Gamma^\theta_{r\theta} T^\theta_\theta - \Gamma^\phi_{r\phi} T^\phi_\phi \\
&= \partial_r \rho(r) + \Gamma^t_{rt} p(r) + \Gamma^r_{rr} p(r) + \Gamma^\theta_{r\theta} p(r) + \Gamma^\phi_{r\phi} p(r) + \Gamma^t_{rt} \rho(r) - \Gamma^r_{rr} p(r) - \Gamma^\theta_{r\theta} p(r) - \Gamma^\phi_{r\phi} p(r) \\
&= \partial_r \rho(r) + \Gamma^t_{rt} p(r) + \Gamma^t_{rt} \rho(r) \\
&= \partial_r \rho(r) + [p(r) + \rho(r)] \partial_r \alpha = 0.
\end{aligned} \tag{16}$$

If we can get $\rho \to \rho(p(r))$ then we could integrate over r. A simple and semi-realistic model of a star comes from assuming that the fluid is incompressible so we make the choice that the density is constant out to the surface of the star

$$\rho \equiv \rho_* = \text{constant}. \tag{17}$$

This assumption is not very realistic but it holds for small objects. Note that the derivative of p with respect to ρ, which is the speed of sound, is not well-defined in this case because ρ does not vary.

$$\frac{\partial p}{\partial \rho} \approx v_s^2 = ? \tag{18}$$

This is the most serious flaw in this assumption. It's not physical but it is a solution to the equation. With this substitution, we can try to integrate

$$\partial_r p + (\rho + p) \partial_r \alpha = 0. \tag{19}$$

By differentiating the substitution made earlier

$$\begin{aligned}
e^{-2\beta} &= 1 - \frac{2GM}{r} \\
-2\partial_r \beta e^{-2\beta} &= -\frac{2G\partial_r m}{r} + \frac{2Gm}{r^2}
\end{aligned} \tag{20}$$

and substituting this into equation 3a but now with a source we find

$$\begin{aligned}
G^t_t &= -\frac{1}{r^2}e^{-2\beta}\left(2r\partial_r\beta - 1 + e^{2\beta}\right) \\
&= -\frac{2}{r}e^{-2\beta}\partial_r\beta + \frac{1}{r^2}e^{-2\beta} - \frac{1}{r^2} \\
&= \frac{1}{r}\left(-\frac{2G\partial_r m}{r} + \frac{2Gm}{r^2}\right) + \frac{1}{r^2}e^{-2\beta} - \frac{1}{r^2} \\
&= -\frac{2G\partial_r m}{r^2} + \frac{2Gm}{r^3} + \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right) - \frac{1}{r^2} \\
&= -\frac{2G\partial_r m}{r^2} + \frac{2Gm}{r^3} + \frac{1}{r^2} - \frac{2GM}{r^3} - \frac{1}{r^2} \\
&= -\frac{2G\partial_r m}{r^2} = -8\pi G\rho_* \qquad (21)
\end{aligned}$$

giving us the relation

$$\frac{dm}{dr} = 4\pi r^2\rho_* \qquad (22)$$

which can be integrated to find an equation for m

$$m(r) = 4\pi\int \rho_* r^2 dr. \qquad (23)$$

This is not a proper integral. It's not an integral over a 3-volume. We could replace this by the usual spherical volume element but there should be a $\sqrt{g_{rr}}$ in here to make this a proper integral. What happened here is that the mass that formed this thing is bigger than the mass that we are in orbit around because the binding energy has not been considered here. However, we need to solve the integral, regardless.

Evaluating equation 3b with a source

$$G^r_r = \frac{1}{r^2}e^{-2\beta}\left(2r\partial_r\alpha + 1 - e^{2\beta}\right) = 8\pi Gp \qquad (24)$$

we find

$$\frac{\partial\alpha}{\partial r} = \frac{\left(8\pi Gr^2p + 1\right)e^{2\beta} - 1}{2r} \qquad (25)$$

We can ask: what is the relationship between m and β? To answer this, we look back at the substitution made earlier and find

$$m(r) = \frac{r}{2G}\left(1 - e^{-2\beta}\right). \qquad (26)$$

We can rearrange Eqn. 19 to get

$$\frac{\partial\alpha}{\partial r} = \frac{-\frac{\partial p}{\partial r}}{(\rho + p)} = \frac{8\pi Gr^2p + 2Gm/r}{2rc^2\left(1 - \frac{2Gm}{rc^2}\right)} \qquad (27)$$

which finally gives

$$-\frac{\frac{\partial p}{\partial r}}{(\rho+p)} = \frac{G\left(m(r)+4\pi r^3 p\right)}{r^2c^2\left(1-\frac{2Gm(r)}{rc^2}\right)} \tag{28}$$

As $c \to \infty$ this equation becomes

$$\frac{\partial p}{\partial r} = -\rho\frac{Gm}{r^2} \tag{29}$$

where we can recognize Gm/r^2 as the acceleration due to gravity and we wind up with

$$dp = -\rho g dr. \tag{30}$$

This process ultimately leaves us with the **Tolman-Oppenheimer-Volkoff equation**, the equation of hydrostatic equilibrium

$$\frac{dp}{dr} = -\frac{G(\rho+p)[m(r)+2\pi r^3 p]}{r^2c^2-2Gm(r)} \tag{31}$$

which we can rewrite using $m = (4/3)\pi r^3\rho_*$ and separate into an integrable form

$$\frac{-dp}{(\rho_*+\frac{p}{c^2})(\rho_*+\frac{p}{3c^2})} = 4\pi\frac{drr}{1-8\pi G\rho_* r^2/c^2} \tag{32}$$

to obtain a relation between p and R

$$p(r) = \rho_*\left[\frac{R\sqrt{R-2GM/c^2}-\sqrt{R^3-2GMr^2/c^2}}{\sqrt{R^3-2GMr^2c^2}-3R\sqrt{R-2GM/c^2}}\right] \tag{33}$$

where $M = (4/3)\pi R^3\rho_*$. From this we can see that when $r = R$, the pressure is zero. Finally we get the metric component $g^t_t = -e^{2\alpha}$ by integrating Eqn. 25 and find

$$e^\alpha = \frac{3}{2}\left(1-\frac{2GM}{R}\right)^{1/2} - \frac{1}{2}\left(1-\frac{2GMr^2}{R^3}\right)^{1/2}, \quad r < R. \tag{34}$$

The pressure increases near the core of the star, as one would expect. For a star of fixed radius R, the central pressure $p(0)$ will need to be greater than infinity if the mass exceeds

$$M_{\max} = \frac{4}{9G}R. \tag{35}$$

Thus, if we try to squeeze a greater mass than this inside a radius R, general relativity admits no static solutions; a star that shrinks to such a size must inevitably keep shrinking, eventually forming a black hole. Although we derived this result from the assumption that the density is constant, it continues to hold with that assumption considerably weakened; **Buchdahl's theorem** states that any reasonable static, spherically symmetric interior solution has $M < 4R/9G$.