PHZ 6607 Special and General Relativity
October 3, 2018

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1 Scattering in Schwarzschild

Today we will discuss unbound orbits in the Schwarzschild geometry, or scattering off of a Schwarzschild black hole. Previously, we derived the geodesic equation

$$\left(\frac{dr}{d\tau}\right)^2 = c^2\big((E/mc^2)^2 - 1\big) + \frac{2GM}{r} - \frac{L^2}{m^2r^2} + \frac{2GML^2}{m^2r^3c^2},$$

where $E = m(1 - 2GM/rc^2)dt/d\tau$ is the relativistic energy of the particle at infinity and $L = mr^2\sin^2\theta d\phi/d\lambda$ is the angular momentum of the particle. We also defined the variable $u = 1/r$, the angular momentum per unit mass $\tilde{L} = L/m$, and the relativistic energy per unit mass $\tilde{E} = E/mc^2$. Making these substitutions, we found

$$\frac{d^2u}{d\phi^2} = \frac{GM}{\tilde{L}^2} - u + \frac{3GM}{c^2}u^2$$

We defined the impact parameter b by $GM/\tilde{L}^2 = 1/b$. We will now solve this equation to first order in the quantity $3GM/c^2$, so we write

$$u = u_0 + \left(\frac{3GM}{c^2}\right)u_1 + O\Big(\Big(\frac{3GM}{c^2}\Big)^2\Big)$$

1.1 Zeroth order

The zeroth order equation is then

$$\frac{d^2u_0}{d\phi^2} = \frac{1}{b} - u_0.$$

The solution to this equation is

$$u_0 = \tfrac{1}{b}(1 + \varepsilon\cos\phi),$$

where $\varepsilon > 1$ is the eccentricity for a hyperbolic orbit. Recall that

$$\pi/2 < \phi_{\max} = \arccos(-1/\varepsilon) < \pi.$$

Taking the derivative, we get

$$\frac{du_0}{d\phi} = -\frac{\varepsilon}{b}\sin\phi = -\frac{1}{r^2}\frac{dr}{d\phi}.$$

Since $du/d\phi = -(1/r^2)dr/d\phi$, we can now examine what happens at large r, or as $\phi \to \phi_{\max}$. Taking this limit,

$$\lim_{\phi \to \phi_{\max}} \frac{1}{r^2}\frac{dr}{d\phi} = \frac{\epsilon}{b}\sin(\arccos(-1/\varepsilon)) = \frac{\varepsilon}{b}\sqrt{1-(1/\varepsilon)^2} = \frac{\sqrt{\varepsilon^2-1}}{b} = \sqrt{\frac{c^2}{\tilde{L}^2}(\tilde{E}^2-1)},$$

Separately, since in this limit we know that

$$\frac{dr}{d\tau} = \frac{\tilde{L}}{r^2}\frac{dr}{d\phi} = c\sqrt{\tilde{E}^2-1},$$

we can solve for the eccentricity:

$$\begin{aligned}\frac{\sqrt{\varepsilon^2-1}}{b} &= \sqrt{\frac{c^2}{\tilde{L}^2}(\tilde{E}^2-1)} \\ \implies \varepsilon^2 &= 1 + \Big(\frac{\tilde{L}c}{GM}\Big)^2(\tilde{E}^2-1).\end{aligned}$$

This reproduces the non-relativistic result.

2 First order

Given knowledge of u_0, the first order equation is

$$\frac{d^2u_1}{d\phi^2} = -u_1 + \frac{3GM}{c^2}\frac{1}{b^2}(1 + 2\varepsilon\cos\phi + \tfrac{\varepsilon^2}{2}(1+\cos 2\phi)).$$

The solution to this equation is given by

$$u_1 = A + B\phi\cos\phi + c\cos 2\phi,$$

where

$$A = \frac{3GM}{c^2}\frac{1}{b^2}(1+\varepsilon^2/2), \quad B = \frac{3GM}{c^2}\frac{1}{b^2}\varepsilon, \quad C = -\frac{GM}{c^2}\frac{1}{b^2}\frac{\varepsilon^2}{2}.$$

Thus, our solution for $u = 1/r$ to first order is

$$u = u_0 + u_1 = \frac{1}{b}\Big(1 + \frac{GM}{bc^2}(3+2\varepsilon^2) + \varepsilon\cos\big((1-\tfrac{3GM}{bc^2})\phi\big) - \frac{GM}{bc^2}\varepsilon\cos^2\phi\Big).$$

By combining the trigonometric terms and taking an approximation, we arrive at the new deflection angle

$$\phi_{\max} = \Big(1 - \frac{3GM}{bc^2}\Big)^{-1}\arccos\Big(-\frac{1+\frac{2GM}{bc^2}(1+\varepsilon^2)}{\varepsilon}\Big),$$

which is advanced compared to the $\phi_{\max}$ in the Newtonian case.