# PHY 6607 Special and General Relativity (10/26/2018)

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## 1. Geodesics of Schwarzschild

Following from last class, we start with the Lagrangian:

$$\mathcal{L} = \frac{1}{2}m\Big[-c^2\Big(1-\frac{2GM}{rc^2}\Big)\Big(\frac{dt}{d\lambda}\Big)^2 + \frac{(\frac{dr}{d\lambda})^2}{(1-\frac{2GM}{rc^2})} + r^2\Big(\frac{d\theta}{d\lambda} + \sin^2\theta(\frac{d\phi}{d\lambda})^2\Big).$$

Recall that we will also use the following relation:

$$p^\mu p^\nu g_{\mu\nu} = -m^2c^2\,, \tag{1}$$

and for photons (and all other massless particles):

$$p^\mu p^\nu g_{\mu\nu} = 0\,.$$

In addition, $\tau \to 0$. So we can consider:

$$\lim_{m\to 0}\lim_{\tau\to 0}\frac{m}{d\tau^2} \to \frac{1}{d\lambda^2}\,.$$

In some sense, this is not quite satisfactory because this says that $[\lambda] = \mathrm{s}\cdot\mathrm{kg}^{-1/2}$ which is not natural. However, if we choose:

$$P_t = -E \equiv h\nu\,,$$

then $[\lambda] = \mathrm{s}$. What we are doing here is making the numerical value of the mass $m$ hold the dimensionality of the mass $m$.

Let's now look at the effective potential for photons. We will begin by calculating the momenta, and for the moment we will leave mass $m$ in:

$$p_t = -E = -m\Big(1-\frac{2GM}{rc^2}\Big)\frac{dt}{d\lambda}\,,$$

$$p_\phi = L = mr^2\sin^2\theta\frac{d\phi}{d\lambda}\,.$$

Writing out Equation 1 and choosing an equatorial orbit ($\theta = \pi/2$, $\phi = 0$):

$$\frac{-E^2}{c^2(1-\frac{2GM}{rc^2})} + \frac{m^2(\frac{dr}{d\lambda})^2}{(1-\frac{2GM}{rc^2})} + \frac{L^2}{r^2} = -m^2c^2\,.$$

We can simplify this expression by writing in terms of $\tilde{E} = E/m$ and $\tilde{L} = L/m$:

$$\frac{-\tilde{E}^2}{c^2\Big(1-\frac{2GM}{rc^2}\Big)} + \frac{m^2(\frac{dr}{d\lambda})^2}{(1-\frac{2GM}{rc^2})} + \frac{\tilde{L}^2}{r^2} = -\varepsilon c^2$$

where:

$$\varepsilon = \begin{cases} 1 & : \text{timelike} \\ -1 & : \text{spacelike} \\ 0 & : \text{lightlike} \end{cases} .$$

So far, this is the same as last lecture (see previous class notes). However, we will now consider for a photon ($m = 0$, lightlike) and solve for $dr/d\lambda$:

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{\tilde{E}^2}{c^2} - \frac{\tilde{L}^2}{r^2}\left(1 - \frac{2GM}{rc^2}\right). \tag{2}$$

Let's define $V_{eff}^2$ as:

$$V_{eff}^2 = \frac{\tilde{L}^2}{r^2}\left(1 - \frac{2GM}{rc^2}\right).$$

We are interested in the derivative of $V_{eff}^2$ with respect to $r$ is:

$$\frac{dV_{eff}^2}{dr} = \frac{-2\tilde{L}^2}{r^3} - \frac{2\tilde{L}^2}{r^3}\left(-\frac{2GM}{rc^2}\right) + \frac{\tilde{L}^2}{r^2}\frac{2GM}{r^2c^2}\Big),$$

$$\frac{dV_{eff}^2}{dr} = -\frac{2\tilde{L}^2}{r^3}\left(1 - \frac{3GM}{rc^2}\right).$$

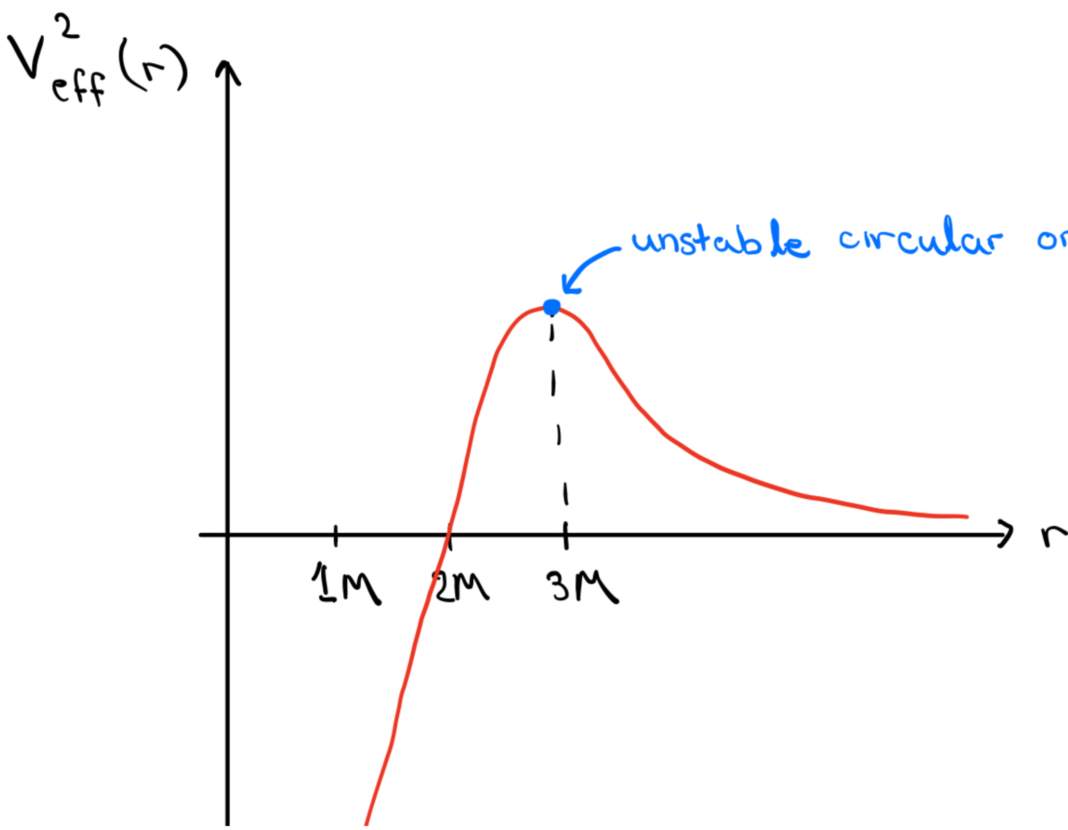


**Figure 1.** Effective potential for a photon in Schwarzschild.

Figure 1 shows a plot of $V_{eff}^2$. For every value of $\tilde{L}$, $V_{eff}^2$ is zero at $r = 2M$ and $V_{eff}^2$ is maximum at $r = 3M$. So, a light ray of energy $E$ coming in will scatter back if $E^2 < V_{eff}^2$. And a light ray can sit in an unstable circular orbit if $E^2 = V_{eff}^2$.

## 2. Tying this into detection of gravitational waves made by LIGO

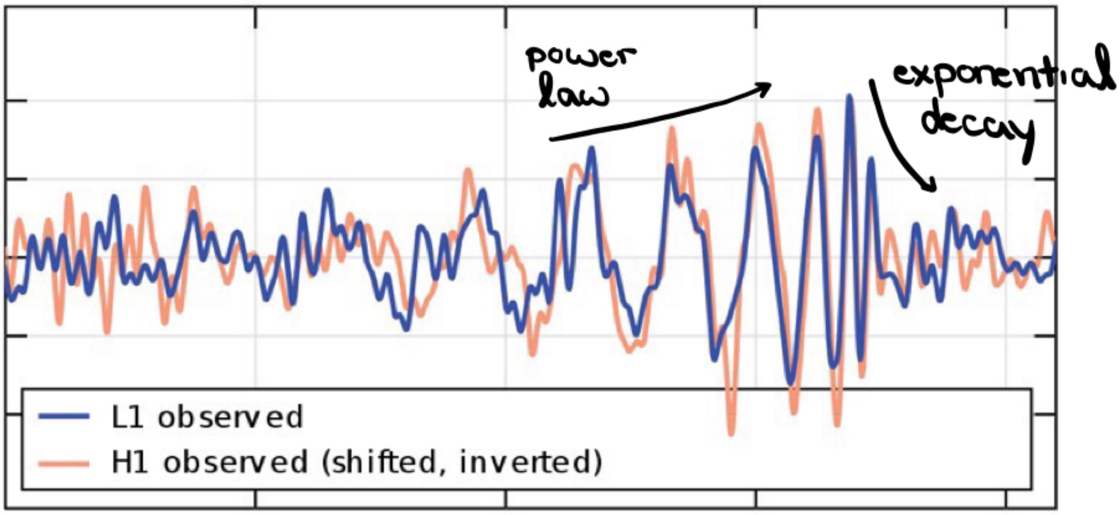


**Figure 2.** Adapted from GW150914 detection paper.

On September 14, 2015, gravitational waves caused by two black holes merging were detected by LIGO (GW150914). After the peak of the signal from the merger, the signal decayed exponentially. The exponential decay can be associated with photons that are sitting near (but not directly at) $r = 3M$, the radius for unstable circular orbit. Remember that this is a dynamical spacetime so the unstable point is actually moving. Photons sitting in the neighborhood of this unstable point will orbit until they leak out. So, the frequency and decay time of the gravitational wave signal can be estimated from knowing this potential $V_{eff}$. The exponential decay is composed of an infinite number of quasinormal modes, which correspond to a pair of points in the complex plane (see Figure 3). Even with the data from GW150914 (and other black black hole events since), we could not deduce information about the quasinormal modes.

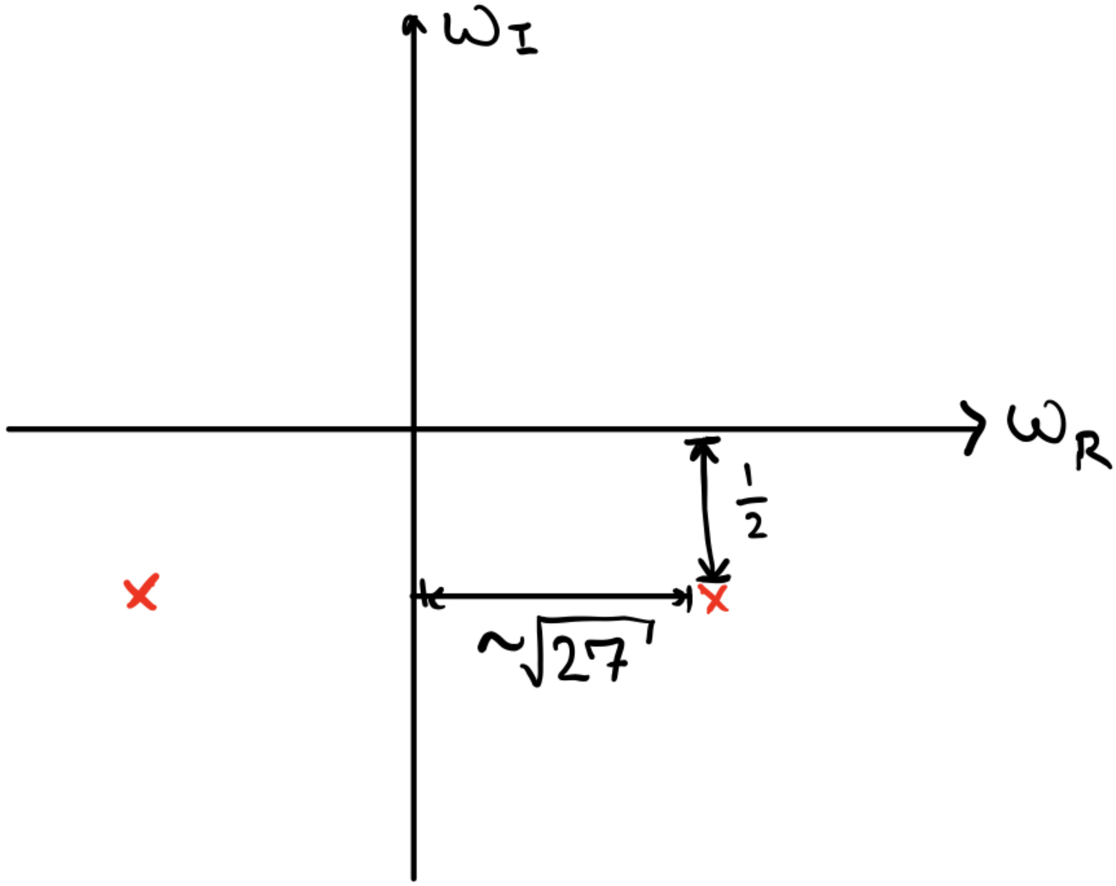


**Figure 3.** Cartoon of the most long lived quasinormal mode (with units of $M$, $R$ surpressed). The real part corresponds to the frequency of the mode, and the imaginary part corresponds to the decay rate. The factor of 1/2 indicates that the mode decays quickly.

## 3. Orbits of photons around black holes

Let's consider the circular orbits. Setting $r = 3M$ and including dimensionality:

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{\tilde{E}^2}{c^2} - \frac{\tilde{L}c^4}{27G^2M^2}\,.$$

So a photon of circular orbit has:

$$\tilde{E} = \frac{1}{\sqrt{27}}\frac{\tilde{L}c^3}{GM}\,.$$

Next time we will discuss geometries in which a black hole has existed forever, in which we have what is called a black hole in the future, and a white hole in the past where it is possible for a photon to exit the white hole. For now, we only consider photons which start outside the black hole and could either bounce off the potential or go in the black hole. If there is no angular momentum $\tilde{L} = 0$, there is no turning point and the photon will just fall straight in. If the photon has low enough energy, then it will not hit the black hole and will turn out at some turning point dependent on the angular momentum. For a fixed angular momentum, the turning point will be further and further away for lower energies. This corresponds to Figure 4.

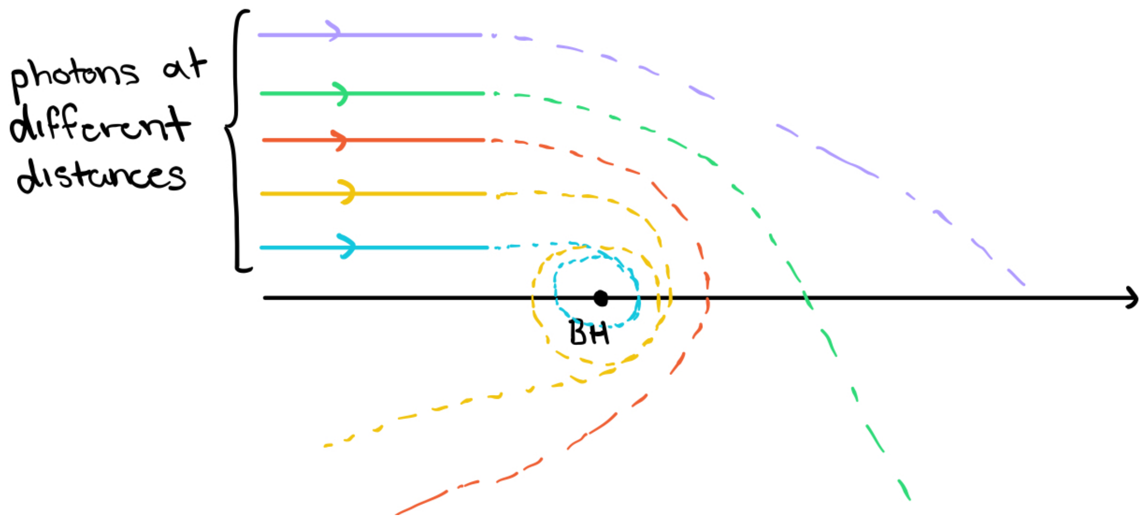


**Figure 4.**

Next, we can look at the scattering angle $\Delta\phi$ (see Figure 5). Returning to Equation 2, we will let $x = L^2/GMr$. The other equation we need is:

$$\tilde{L} = r^2 \frac{d\phi}{d\lambda} \,.$$

It follows then that:

$$\frac{d}{d\lambda} = \frac{\tilde{L}}{r^2}\frac{d}{d\phi} \,,$$

and:

$$d\frac{1}{r} = \frac{GM}{\tilde{L}} dx \,.$$

Substituting this in, we get:

$$\frac{\tilde{L}}{r^2}\frac{dr}{d\phi} = -GM\frac{dx}{d\phi} \,.$$

We will use Equation 2 to solve for $d\phi/dx$:

$$-\frac{GM}{\tilde{L}}\frac{dx}{d\phi} = \frac{1}{\tilde{L}}\Big[\frac{\tilde{E}^2}{c^2} - \Big(\frac{GMx}{\tilde{L}}\Big)^2\Big(1 - \frac{2GM}{c^2\tilde{L}}\frac{GMx}{\tilde{L}}\Big)\Big]^{\frac{1}{2}} \,.$$

Making one more substitution $u = GMx/\tilde{L}$:

$$\frac{du}{d\phi} = \Big[\frac{\tilde{E}^2}{\tilde{L}^2c^2} - \frac{1}{\tilde{L}^2}u^2\Big(1 - \frac{2GM}{\tilde{L}^2c^2}u\Big)\Big]^{\frac{1}{2}} \,.$$

We are interested in the change in angle so we can write:

$$d\phi = du\Big[\frac{\tilde{E}^2}{\tilde{L}^2c^2} - \frac{1}{\tilde{L}^2}u^2\Big(1 - \frac{2GM}{\tilde{L}^2c^2}u\Big)\Big]^{-\frac{1}{2}}$$

If we choose an orbit far so that the deflection is not large, we can treat $D \equiv 2GM/\tilde{L}c^2$ as small:

$$d\phi = du\Big[\frac{\tilde{E}^2}{\tilde{L}^2c^2} - Du\Big)\Big]^{-\frac{1}{2}}$$

We can replace:

$$y = u(1 - Du)\,,$$

and dropping higher order terms:

$$dy = du(1 - Dy) + \mathrm{HOT}\,,$$

$$y^2 = u^2(1 - 2Du) + \mathrm{HOT}\,.$$

We are left with:

$$d\phi = dy(1 + 2Dy)\Big[\frac{\tilde{E}^2}{\tilde{L}^2c^2} - \frac{y}{\tilde{L}^2}\Big)\Big]^{-\frac{1}{2}}$$

Integrating both sides:

$$\Delta\phi = \int dy(1 + 2Dy)\Big[\frac{\tilde{E}^2}{\tilde{L}^2c^2} - \frac{y}{\tilde{L}^2}\Big)\Big]^{-\frac{1}{2}}$$

We want the first term on the r.h.s. in the form of:

$$\int \frac{dz}{\sqrt{1 - z^2}} = \arcsin(z)\,.$$

This is achieved with the substitution of $z = cy/E$. The second term on the r.h.s. is of the form:

$$\int \frac{Dydy}{\sqrt{A - By^2}} = -\frac{2D}{B}\sqrt{A - By^2}\,.$$

So the bending angle is:

$$\Delta\phi = \frac{4D\sqrt{A}}{B}\,,$$

$$\Delta\phi = \frac{4GM\tilde{E}\tilde{L}}{c^3}\,.$$

When we do a similar calculation for a massive particle, the deflection is:

$$\Delta\phi = \frac{6GM\tilde{E}\tilde{L}}{c^3}\,.$$

So a photon follows an orbit which is different from the orbit of any massive particle.

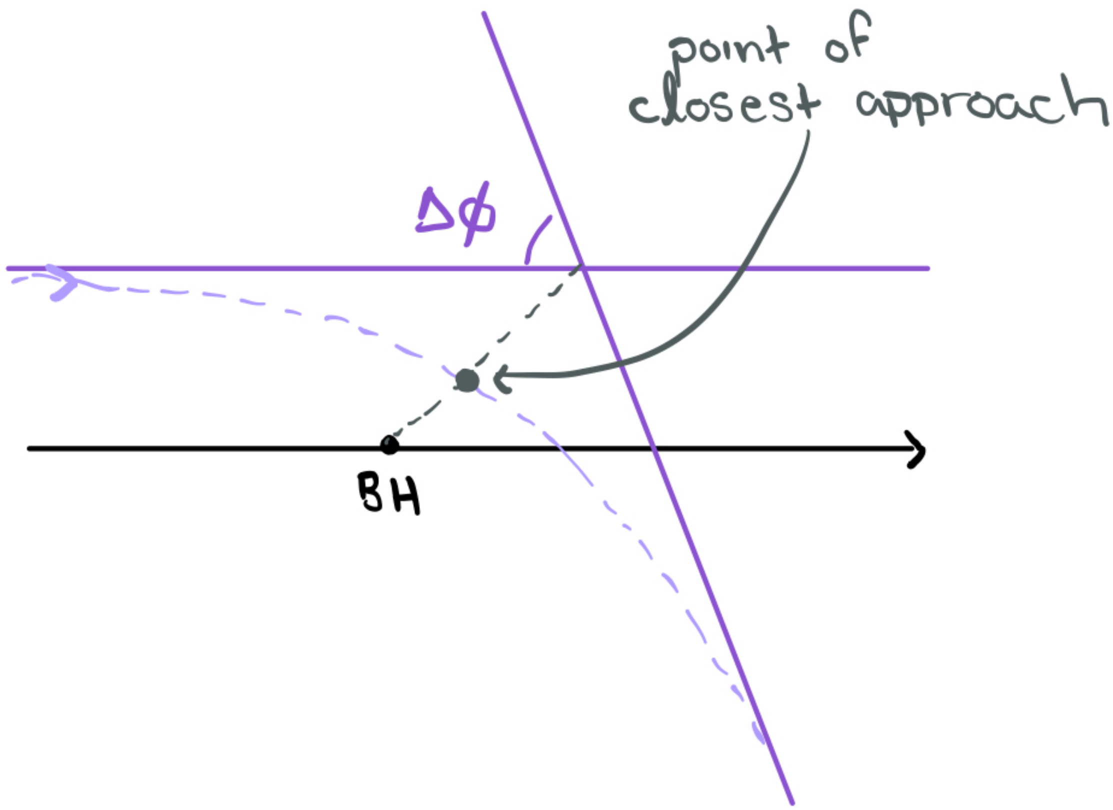


**Figure 5.**