Special and General Relativity PHZ 6607

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Metric in Weak Field limit

We start of by writing the metric for a curved geometry in the weak field limit as

$$g_{\mu\nu} = -(1+2\phi)dt^2 + (1-2\phi)\delta_{ij}dx^i dx^j$$

where the last term in the above expression will give us the spatial part of our proper distance. If we write the metric in spherically symmetric geometry we can see the similarity in these two cases under a limit.

$$ds^2 = \left(1 - \frac{2GM}{Rc^2}\right)dt^2 + \frac{dR^2}{\left(1-\frac{2GM}{Rc^2}\right)} + R^2 d\Omega^2$$

In the spherically symmetric metric, if we evaluate this under the limit that R $\gg GM$, we can see how this metric will end up being the weak field limit, if we simply substitute $\phi = -\frac{GM}{rc^2}$ in the weak field metric. They are still not exactly same and it will give us the spherical symmetric metric under a limit where $R, r \gg 0$. As important as they are, these metrics are just approximations and not exact solutions of the Einstein equations.

The exact solutions give us a slightly different form of the metric i.e.

$$\frac{\left(1-\frac{GM}{2rc^2}\right)^2}{\left(1+\frac{GM}{2rc^2}\right)^2}dt^2 + \left(1+\frac{GM}{2rc^2}\right)^4\left(dr^2 + r^2 d\Omega^2\right)$$

and obtaining the weak field limit is straightforward under the assumption that R $\gg \frac{GM}{c^2}$, the expressions will end up becoming,

$$\begin{aligned} g_{tt} &= \left(1-\frac{GM}{2rc^2}\right)\left(1-\frac{GM}{2rc^2}\right) \\ &= \left(1-\frac{2GM}{rc^2}\right) + O(higher) \\ &= (1+2\Phi) \end{aligned} \qquad \begin{aligned} g_{ij} &= 1 + \frac{4GM}{2rc^2} \\ &= \left(1+\frac{2GM}{rc^2}\right) \\ &= (1-2\Phi) \end{aligned}$$

which gives us back the weak field metric.

Difference in Static and Stationary

A notable feature of the exact solution is that it is a static solution i.e. nothing changes in the equation* as a function of time. The difference between stationary and static metric is that a stationary metric will have off diagonal terms in the metric which will being in a dependence for the coordinates on time. As against in a static metric, there is no dependence of time. Having $\frac{\partial}{\partial t}$ as a Killing vector only guarantees a stationary condition and not a static condition.

An exercise that we can do here is to get the spherically symmetric metric from this exact solution by matching the co-efficients of the appropriate terms.

Comparing just the co-efficient of the angular part we get,

$$R(r)^2 = \left(1 + \frac{GM}{2rc^2}\right)^4 r^2$$

$$R(r) = \left(1 + \frac{GM}{2rc^2}\right)^2 r$$

$$dR = dr\left(1 - \left(\frac{GM}{2rc^2}\right)^2\right)$$

and upon re-substituting this in the Spherically Symmetric Metric, we find that all coefficients match. What this tells us is the R in the Spherically Symmetric metric is not the same as the r in the weak field metric.

We can in fact solve for one in terms of the other by using the radial term in the metric by integrating

$$\frac{dR}{\sqrt{1 - \frac{2GM}{Rc^2}}} = dr\left(1 + \frac{GM}{2rc^2}\right)^2$$

and using the expression from angular part to evaluate the Constant of Integration.

Rigorous derivation of the metric

We started of working with the metrics directly without actually deriving it. However, we can derive them starting from a very general form,

$$ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2 e^{2\gamma(r,t)}d\Omega^2$$

We can rid off all the time dependence as $\frac{\partial}{\partial x}$ is still a Killing Vector in our system. We can now try a solution by starting with $R = r^2 e^{2\gamma}$ to get,

$$ds^2 = -e^{2\tilde{\alpha}(R)}dt^2 + e^{2\tilde{\beta}(R)}dR^2 + R^2 d\Omega^2$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ account for the transformations that have happened. In this metric we evaluate the Ricci Tensor and set it to zero to get conditions on these two functions.

Individually making the components of R_{tt}, R_{rr} vanish will give us the following conditions,

$$0 = e^{(\beta-\alpha)}R_{tt} + R_{rr} = \frac{2}{r}(\partial_r\alpha + \partial_r\beta)$$

which sets $\alpha = -\beta$ and if we look at the $R_{\theta\theta} = 0$, we get $e^{2\alpha}(2r\partial_r\alpha + 1) = 1$ which give us the relation $e^{2\alpha} = 1 - \frac{R_s}{r}$ making the metric in the end to give,

$$ds^2 = -\left(1 - \frac{R_s}{r}\right)dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

where the R_s will come from the Schwarzschild Radius and will be set equal to $R_s = 2GM$

Singularities

It is evident that both metrics that we have singularities in two places. In the Schwarzschild metric, we see a singularity at $R = \frac{2GM}{c^2}$ and another at $R = 0$. This is something worth investigating as singularities can arise purely from the choice of coordinates and hence we need a stronger indicator. To see if there is actually something wrong with the geometry, we want to see a coordinate independent quantity like the Ricci Tensor. To be robust about the measurement, we can construct many other higher order scalars like this and use that to see the behaviour at the two singularities we have discussed before.

We can look at

$$R_{\mu\nu\sigma\rho} \propto \frac{M}{r^3} \qquad\qquad R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} \propto \frac{M^2}{r^6}$$

which are well behaved at $r = 2M$ but are still singular at $r = 0$. This tells us a lot about the nature if these singularities as one of the singularity$r = 0$ is a true singularity of the geometry, whereas $r = 2M$ is just a relic of the coordinate system that we have chosen. As these scalar quantities are independent of the choice of a frame, we can assert that $r = 0$ is a a true singularity in any coordinate system.