# PHZ 6607: Special and General Relativity I
# Lecture Notes

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In this lecture we discuss a little bit more the Equivalence Principle and then we review briefly a broad family of alternative theories.

## Extra Comments regarding the Equivalence Principle

One way the Equivalence Principle is usually expressed is by stating the equivalence between inertial and gravitational mass. Newton's second Law says that:

$$\mathbf{F} \,=\, m_I \,\mathbf{a}$$

where $m_I$ is the inertial mass, while Newton's Law of Gravitation says that the force of gravitational attraction from a second object is:

$$\mathbf{F} \,=\, -G\,\frac{m_G m_2}{r^2}\,\hat{\mathbf{r}} \,=\, m_G\,\mathbf{g}$$

where $m_G$ ans $m_2$ are the gravitational masses of the objects. Thus this version of the Equivalence Principle states that

$$m_I \,=\, m_G.$$

And there are experimental tests of this statement. Let's think of the situation in Figure 1. Matter falls towards a central point, which could be the Sun or a solid object. The material arrives at the central region with some kinetic energy and this energy becomes heat and is radiated away. So the Sun that we are orbiting as the Earth, is the energy that was stored in these chunks of matter, minus the gravitational binding energy which has been radiated away. Now actually the Earth and the Sun orbit around their common center of mass which would be in a slightly different place if the binding energy was not radiated away. So this equivalence is closely related to the fact that gravitational energy gravitates. If gravitational energy did not gravitate, then getting rid of it would not change the gravitational attraction. But the equivalence between mass and any kind of energy is something that can be tested, and from these measurements we find that the equivalence to at least one part in $10^{15}$. Experiments use bodies of different composition and the Earth as the attractor, so one gets a 24h-periodic signal. Generally the signal is buried in the noise so one needs very careful data analysis to extract the relevant information.

## Alternative Theories

We concentrate basically here one class of theories, namely the scalar-tensor (ST) theories, since many considerations arising in this discussion can be applied to other classes of alternative theories.

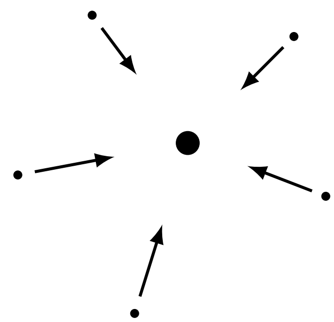

Figure 1: Matter falling toward some central region

In GR, we write the action as:

$$S = S_G + S_m$$

where $S_G$ is the component from gravity (with the scalar Lagrangian density equal to the Ricci scalar R) and $S_m$ the component from matter.

In ST theories we have:

$$S = S_{G,\lambda} + S_m + S_\lambda$$

where the gravitational component includes the scalar field (in the form $f(\lambda(x^\mu)) \cdot R$) and the pure-scalar piece would be

$$S_\lambda = \frac{1}{2} \nabla_\mu \lambda \, \nabla^\mu \lambda - V(\lambda).$$

The ST action changes the equations of motions and the scalar field seems to violate the Equivalence principle since it couples directly to the Ricci tensor instead of the metric. Observations show that the dependence on the $\lambda(x^\mu)$ must be weak up to galactic or even cosmic scales.

It turns out that by fine-tuning the parameters of these theories one can avoid the need for dark matter on the scale of galaxies. But generally these parameters do not give consistent results for dark matter at larger scales (e.g. superclusters or cosmic scales).

By rescaling the metric and changing variables, this theory "resembles" GR:

$$\sqrt{g} \, f(\lambda) \, R \to \sqrt{\tilde{g}} \, \tilde{R} + ...$$

with the extra terms on the right hand side being a new $S'_\lambda$ term. But this is not just a coordinate transformation, since we are changing the fields. Matter action also will change because it generally includes the metric. Finally, the geodesics will change too, since we are changing the geometry. The last two imply that the Stress Energy tensor might change.

Until now most of the test of GR are in the weak field, so if $\lambda(x^\mu)$ changes on the scale of several Mpc, then we don't have many constrains on it.

One vestion of ST theory is the Brans-Dicke theory which involves a dimensionless parameter $\omega$, that is not determined a priori. As $\omega$ goes to infinity the the theory becomes indistinguishable from GR. Measurements of Shapiro time delay using the Cassini satellite indicate a minimum value of $\omega_{min} \approx 4 \cdot 10^4$, while current cosmological tests using data from the Planck Mission give $\omega_{min} \approx 890$.

### $f(R)$ theories (e.g. $R^2$)

We turn our attention to the so called $f(R)$ theories. This belongs into a general class of (generically) Higher Order theories like $R^{\mu\nu}R_{\mu\nu}$ , $\Box$ R , $R^{\mu\nu}{}_{\rho\sigma}R_{\mu\nu}{}^{\rho\sigma}$, .... These theories are generically unstable.

There is another version of these theories which is not higher order. We start from a basic fact from topology. The number of faces minus the number of edges, plus the number of vertices ($F - E + V$) for a complete triangulation of a surface is a constant independent of the particular triangulation one might choose. This is what we call the Euler characteristic $\chi$. For 2D surfaces the one can actually compute $\chi$ using the Ricci

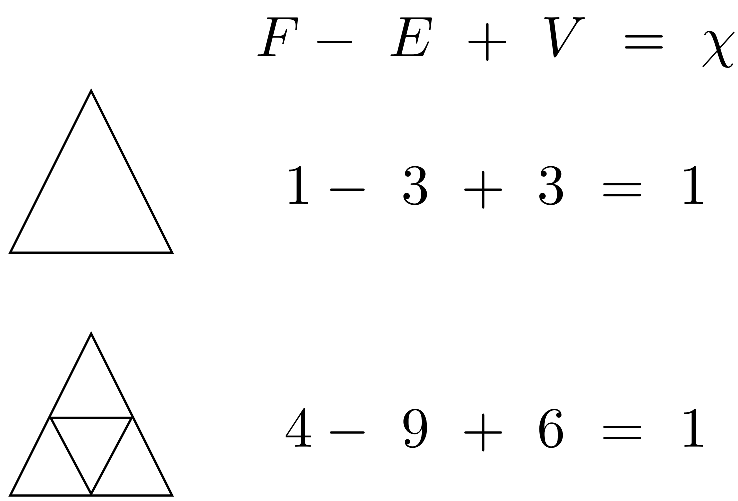


Figure 2: There is a characteristic number common for every triangulation of a particular surface.This number is the Euler characteristic $\chi$.

scalar from:

$$\chi = \int R \sqrt{g}\, \mathrm{d}^2x$$

And there exists an equivalent of $\chi$ for higher dimensional surfaces (Chern–Gauss–Bonnet theorem) and it is given by:

$$\alpha R + \beta R^2 + \gamma R \cdot R + \epsilon \,\Box R + \eta R : R + ...$$

Now using this expression as an action, one can show that we get 2nd order equations of motion, despite the fact that individual pieces are generally of 4th order. This is Lovelock Theory of Gravity.

### Propagation of Gravitational Waves and Alternative theories

Let's consider $g = \eta + \epsilon\, h$ where h is small. Then at some appropriate coordinates (Transverse Traceless gauge, or TT gauge) the perturbed Einstein's Equations become:

$$\Box\, \bar{h}_{\mu\nu} = -16\, \pi\, G\, T_{\mu\nu}$$

The tensor for a wave propagating in the $\hat{z}$ direction is :

$$\begin{pmatrix} h_+ + s_1 & h_\times & v_1 \\ h_\times & -h_+ + s_1 & v_2 \\ v_1 & v_2 & s_2 \end{pmatrix}$$

Gravitational Waves (GW) have two tensor polarizations in GR ($h_+$ and $h_\times$) which appear in different positions in the perturbation tensor than possible vector polarizations predicted in some alternative theories (e.g. massive graviton theories), and thus they interact differently with a gravitational wave detector. Current GW experiments have recently published constrains on these other polarizations. Restricting these modes is one way of restricting the mass of the graviton. The same is true for scalar polarizations ($s_1$ and $s_2$ ; $s_1$ is called the breathing mode).

### $ds^2$ in the regime of small perturbations

Returning back to Einstein's Equations, we have already deduced in a previous lecture that

$$h_{00} \;=\; -2\,\phi$$

as the relation that connects Newtonian Gravity and small perturbations in GR. Now in the TT gauge:

$$\bar{h}_{\mu\nu} \;=\; h_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, h^{\sigma}{}_{\sigma}$$

and thus (using the fact that $g^{\mu\nu}\, g_{\mu\nu} \;=\; \delta^{\nu}{}_{\nu} \;=\; 4$):

$$\bar{h}^{\sigma}{}_{\sigma} \;=\; -h^{\sigma}{}_{\sigma}$$

Also the 0-th components of the equation above give:

$$\bar{h}_{00} \;=\; h_{00} - \frac{1}{2}\,\eta_{00}\, h^{\sigma}{}_{\sigma}$$

Now if the only nonzero component of $\bar{h}_{\mu\nu}$ is $\bar{h}_{00}$, we get:

$$\bar{h}^{\sigma}{}_{\sigma} \;=\; g^{00}\,\bar{h}_{00} = -\bar{h}_{00}$$

and therefore $\bar{h}_{00} \;=\; 2\,h_{00} = -\,4\,\phi$.

Now looking at the spacial components we find:

$$h_{ij} \;=\; \bar{h}_{ij} \;-\; \frac{1}{2}\,\eta_{ij}\,\bar{h}^{\sigma}{}_{\sigma} \;=\eta_{ij}\;2\,\phi$$

Combining these results we get:

$$ds^2 \;=\; -(1-2\phi)\;dt^2 +\;(1+2\phi)(dx^2+dy^2+dz^2)$$

As we will see in the following lectures, this equation turns out to be important in our discussions of approximations to Schwarzschild Geometry.