Special and General Relativity Notes

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Wednesday, October 10th

1 Perfect Fluids

For a perfect fluid in the rest frame we know $T^{\mu\nu} = (\rho, p, p, p)$ where the stress in each direction is the same and that stress is the pressure. What reduces to this in the rest frame is:

$$\boxed{T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U^\mu U^\nu + p g_{\mu\nu}} \tag{1}$$

where U^μ is the fluid four-velocity. In this equation pressure will be suppressed by the $1/c^2$ term. We want to know the strong energy equation. In order to find this we look at Raychaudhuri's Equation.

1.1 Raychaudhuri's Equation

U is a field inside a volume. **Expansion** is give as:

$$\theta = \nabla_\mu U^\mu \tag{2}$$

"The expansion describes the change in volume of a sphere of test particles centered on our geodesic" (Carrol, pg. 460). Let $B_{\nu\mu} = \nabla_\nu U_\mu$ where the trace of $B_{\nu\mu}$ is the expansion. With $\nabla_\nu (U^\mu U_\mu) = 0$ since $U^\mu U_\mu = -c^2$ because the tangent field is normalized. This means it also obeys another equation:

$$U^\nu \nabla_\nu U_\mu = U^\nu B_{\nu\mu} = 0$$

The **shear** is related to the off diagonal of the stress tensor. It is symmetric and traceless and is given as:

$$\sigma_{\mu\nu} = \nabla_{(\nu} U_{\mu)} = B_{(\mu\nu)} - \frac{1}{3}\theta P_{\mu\nu} \tag{3}$$

where $P_{\mu\nu}$ is the projection tensor from 4D to some spatial slice. On the other hand the **rotation** is an antisymmetric tensor give by:

$$\omega_{\mu\nu} = -\nabla_{[\nu} U_{\mu]} = B_{[\mu\nu]} \tag{4}$$

Since a tensor is made out of its antisymmetric and symmetric parts added together we can easily see that:

$$\boxed{B_{\mu\nu} = \frac{1}{3} P_{\mu\nu}\theta + \sigma_{\mu\nu} + \omega_{\mu\nu}} \tag{5}$$

Taking the trace of the covariant derivative of $B_{\mu\nu}$ gives the **Raychaudhuri's equation**:

$$\boxed{\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}U^\mu U^\nu} \tag{6}$$

in this $d\theta/d\tau$ is a scalar. Only the term $R_{\mu\nu}U^\mu U^\nu$ has geometery, the others do not.

In a perfect fluid without sheer and without rotation the Raychaudhuri's equation becomes[1]:

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - R_{\mu\nu}U^\mu U^\nu$$

Now consider a geodesic which starts out parallel, meaning $\theta = 0$. This leaves:

$$\frac{d\theta}{d\tau} = -R_{\mu\nu}U^\mu U^\nu$$

Combining Einstein's equation with the Strong Energy Condition gives us:

$$R_{\mu\nu}U^\mu U^\nu = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)U^\mu U^\nu$$

$$\boxed{R^{\mu\nu} = \frac{8\pi G}{c^4}\left(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T^\sigma{}_\sigma\right)} \tag{7}$$

Using $T^\mu{}_\mu = -\rho c^2 + 3p$ this leaves the equation:

$$R^{\mu\nu} = \frac{8\pi G}{c^4}\left[\left(\rho + \frac{p}{c^2}\right)U^\mu U^\nu + \frac{1}{2}g^{\mu\nu}(\rho - p)\right]$$

Looking at

$$R_{\mu\nu}U^\mu U^\nu$$

in the rest frame:

$$\begin{aligned} R_{00}U^0U^0 &= \frac{8\pi G}{c^4}c^4\left(\rho + \frac{3p}{c^2}\right) \\ &= \frac{8\pi G}{c^2}(\rho c^2 + 3p) \end{aligned}$$

Under strong energy condition: $\rho + 3p \geq 0$.

1. Matter causes geodesics to converge even if we start as diverging, eventually they will converge. *If the geodesic starts out parallel with matter in between, θ will have to decrease and thus converge (which is the motivation for singularity theorems). This means that θ will become negative even for light.*

[1] *Side Note:* When in Minkawski-Space $R_{\mu\nu} = 0$ leaving this as just

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2$$

which would allow the determination of when the geodesics cross.

2. Pressure adds to the effect of matter in the form of density. *Pressure pulls geodesics apart. Gravity from a brick is lesser than the same density of brick made from Hydrogen as the gas will have pressure contribution as well. An equation of state is needed to know how ρ and p relate to eachother in order to solve.*

3. $\nabla_\mu T^{\mu\nu} = 0$ is not enough to solve the equation, but it gives:

$$\frac{dp}{dr} = -\frac{G}{r^2}\frac{\left(\rho + \frac{p}{c^2}\right)\left[m(r) + \frac{4\pi r}{c^2}\right]}{1 - \frac{2Gm(r)}{rc^2}}$$

   where

$$m(r) = 4\pi \int_0^r dr' \rho(r')^2$$

   In hydrostatic equilibrium this gives:

$$\frac{dp}{dr} = -\frac{G}{r^2}\rho m = -g\rho$$

   where $g = Gm/r^2$ *Every relativistic correction causes the pressure gradient to increase.*

1.2 Equivalent Principle

An elevator is a good example of this. Extremely locally gravity and acceleration are equivalent. Two particles in a falling elevator would move toward eachother.

Four other forms of the equivalent principle that have different implications:

1. Our laws of physics are generally covariant.
2. Gravity is the curvature of space-time.
3. Nothing else resembles gravity. *Dark matter shows this is not necessarily true.*
4. Matter doesn't couple to the Riemann Tensor: $\nabla_\mu g^{\mu\nu} \nabla_\nu \phi = 0$. *This is hard to test but so far it is consistent with observations.*