Oct 5th: Stress Energy Tensor

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1 Stress Energy Tensor

We want to find the analogous co-variant form of the Newtonain scalar field Equation: $\nabla^2\phi = 4\pi G\rho$. We choose to work in the slow-motion, weak-field, static metric tensor case. We begin by forming the relation $G_{\mu\nu} = \kappa T_{\mu\nu}$. Recall that the trace tells us: $G^{\mu}_{\mu} = -R^{\mu}_{\mu} = \kappa T^{\mu}_{\mu}$. Therefore, we can say something about the Ricci tensor, namely, $R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$

1.1 Point Particle

For a point particle, we write the stress energy tensor as:

$$T^{\mu\nu} = \left(\frac{m\delta^3(x^\mu - z^\mu(x))}{\sqrt{g}\dot{t}}\right)U^\mu U^\nu \tag{1}$$

Where $\dot{t}$ is $\frac{dt}{d\tau}$ along the proper path. We call the factor in the parenthesis above as ρ so as to write the above equation in a neat form: $T^{\mu}_{\nu} = \rho U^\mu U_\nu$.

1.2 Difference between $T_{\mu\nu}, T^{\mu}_{\nu}, T^{\mu\nu}$

We point out that the dimensions of the stress energy tensor are different based on it's tensor type. That is, T^{μ}_{ν} and $T^{\mu\nu}$ have different dimensions. For example, T^{μ}_{ν} has dimensions of energy density. So when we calculate the trace of T as $Trace(T^{\mu}_{\mu}) = T^0_0 + T^1_1 + T^2_2 + T^3_3$, each term on the right hand side must each have the dimension of energy density.

In general, the 4-velocity is given by $U^\mu = (c\dot{t}, \dot{x}, \dot{y}, \dot{z})$. To lowest order, $U^0 = \dot{t} \sim 1$. That is, in the rest frame of the particle $U^\mu = (1, 0, 0, 0)$. So, $T^{00} = \rho\dot{t}\dot{t}$ and $T^{11} = \rho\dot{x}\dot{x}$ where the signs on the right hand sides are positive. Now, notice we can write $T^0_0 = -\rho c^2\dot{t}^2$. Similarily, $T^1_1 = \rho\dot{x}\dot{x} = \rho v_x^2$. Finally $T_{00} = \rho c^4\dot{t}^2$. The point to take away from this is that T does not always have the same dimensions w.r.t number of indices that are up or down. Finally, we can now calcualte $R_{00} = T_{00} - \frac{1}{2}g_{00}T = \frac{\kappa}{2}\rho c^4$, where $g_{00} = -c^2 - 2\phi$.

We write the Ricci tensor, as a contraction of the Riemann Curvature tensor, in terms of derivatives of the metric tensor as,

$$R_{00} = R^i_{0i0} = \partial_i\left(\frac{1}{2}g^{i\lambda}\left(\partial_t g_{\lambda 0} + \partial_t g_{0\lambda} - \partial_\lambda g_{00}\right)\right) \tag{2}$$

Notice, in our simplified case, we are working with a static metric tensor, i.e. all partial derivatives of the metric tensor w.r.t. coordinate time t, will vanish. This leads us with

$$R_{00} = -\frac{1}{2}\partial_i g^{ij}\partial_j g_{00} = -\frac{1}{2}\nabla^2 g_{00} = \kappa\frac{1}{2}\rho c^4 \tag{3}$$

where $\nabla^2\phi = \frac{\kappa}{2}c^4\rho$, so $\kappa = \frac{8\pi G}{c^4}$. This finishes our analogous calculation of the Newtonian scalar field equation with the following remarks: $G_{00} = \frac{8\pi G}{c^4}T_{00}$ and $G_{ij} = \kappa T_{ij}$.

1.3 Einstein's Equations

We write the Einstein's Equations as

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{4}$$

We notice the above equation is symmetric w.r.t. the μ, ν indicies. This suggests that there are a total of 10 independent, highly non-linear, equations subject to the condition that $\nabla_\mu G^{\mu\nu} = 0$ which gives us a total of 4 conditions. Recall that we have plenty of gauge freedom. So up to a coordinate transformation or gauge transformation, choosing 4 coordinates is enough to solve these equations.

1.3.1 Parallel with Maxwell's Equations

Maxwell	GR
A_μ	$\mathrm{g}_{\mu\nu}$
$\mathrm{F}_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$	$\mathrm{R}^\rho_{\mu\nu\sigma} \sim \partial g_{\mu\nu}$
$\nabla_\mu F^{\mu\nu} = -J^\nu$	$\mathrm{G}_{\mu\nu} = \kappa T_{\mu\nu}$
$\epsilon^{\mu\nu\rho\sigma}_\nu F_{\rho\sigma} = 0$	$\epsilon^{\tau\mu\nu\rho} R_{\mu\nu\rho\sigma} = 0$
$\nabla_\nu \nabla_\mu F^{\mu\nu} = 0$	$\nabla_\mu R\nu\sigma\rho\tau + \nabla_\sigma R_{\mu\nu\rho\tau} + \nabla_\nu R_{\sigma\mu\rho\tau} = 0$