PHZ 6607 Special and General Relativity
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1 Guage transformation vs coordinate transformation

As a brief aside, we'll explicate the differences between a coordinate transformation and a gauge transformation.

A coordinate transformation is defined by writing primed coordinates in terms of unprimed coordinates $x^{\mu'}(x^\mu)$. A coordinate transformation of a tensor is a linear transformation done by the matrices ${\Lambda^{\mu'}}_\mu = \partial x^{\mu'}/\partial x^\mu$ and ${\Lambda^\mu}_{\mu'} = \partial x^\mu/\partial x^{\mu'}$. For example, the transformation of the metric is given by

$$g_{\mu\nu} \to g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} g_{\mu\nu} \frac{\partial x^\nu}{\partial x^{\nu'}}.$$

While clearly the components of $g_{\mu\nu}$ change under a coordinate transformation, the tensor (as a 2-form) is identical in both coordinates because the basis changes: $dx^{\mu'} = d(x^\mu(x^{\mu'}))$. In other words,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu'\nu'} dx^{\mu'} dx^{\nu'}.$$

Constrastly, for a gauge transformation is defined by the pullback of a diffeomorphism. Vector fields generate diffeomorphisms by "moving" points on the manifold along the vector field. As an example, we take a diffeomorphism $\phi : M \to M$ generated by the vector field ξ. An infinitesimal gauge transformation of the metric given by this diffeomorphism parameterized by ϵ is given by

$$\begin{aligned}
ds'^2 &= \phi^* ds^2 = (\phi^* g_{\mu\nu})(\phi^* dx^\mu)(\phi^* dx^\nu) \\
&= g_{\mu\nu}(x^\mu + \epsilon\xi^\mu(x^\nu))(dx^\mu + \epsilon\frac{\partial \xi^\mu}{\partial x^\alpha} dx^\alpha)(dx^\nu + \epsilon\frac{\partial \xi^\nu}{\partial x^\beta} dx^\beta) \\
&= \Big(g_{\mu\nu}(x^\mu) + \frac{\partial g_{\mu\nu}}{\partial x^\gamma}\epsilon\xi^\gamma\Big) dx^\mu dx^\nu + \epsilon g_{\mu\nu}\frac{\partial \xi^\nu}{\partial x^\beta} dx^\mu dx^\beta + \epsilon g_{\mu\nu}\frac{\partial \xi^\mu}{\partial x^\alpha} dx^\alpha dx^\nu + \cdots \\
&= \Big(g_{\mu\nu} + \epsilon\Big(\xi^\gamma \frac{\partial g_{\mu\nu}}{\partial x^\gamma} + g_{\alpha\nu}\frac{\partial \xi^\alpha}{\partial x^\mu} + g_{\mu\beta}\frac{\partial \xi^\beta}{\partial x^\nu}\Big) + \cdots\Big) dx^\mu dx^\nu \\
&= \Big(g_{\mu\nu} + \epsilon\mathcal{L}_\xi g_{\mu\nu} + \cdots\Big) dx^\mu dx^\nu
\end{aligned}$$

This containes derivatives of the metric, which does not happen in a coordinate transformation. Also

$$\begin{aligned}
\mathcal{L}_\xi g_{\mu\nu} &= \xi^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + g_{\alpha\nu}\frac{\partial \xi^\alpha}{\partial x^\mu} + g_{\mu\alpha}\frac{\partial \xi^\alpha}{\partial x^\nu} \\
&= \xi^\alpha \nabla_a g_{\mu\nu} + g_{\alpha\nu}\nabla_\mu \xi^\alpha + g_{\mu\alpha}\nabla_\nu \xi^\alpha \\
&= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu
\end{aligned}$$

which is a diffeomorphism.

Similarly, for an arbitrary vector field, we can show that under the same gauge transformation,

$$\begin{aligned} V_a dx^a \to & \Big(V_a + \epsilon\Big(\xi^b \frac{\partial V_a}{\partial x^b} + V_b \frac{\partial \xi^b}{\partial x^a}\Big)\Big) dx^a \\ = & \Big(V_a + \epsilon \mathcal{L}_\xi V_a\Big) dx^a \end{aligned}$$

Note that we needed vector fields in order to generate the isomorphism. We didn't need vector fields for coordinate transformations.

Carroll loosely compares the difference between gauge transformations and coordinate transformations as the difference between active and passive transformations.

2 Gravitational Motion

Is there a way to associate gravity with geodesic motion in a curved space?

In Newtonian gravity, we have

$$a_i = -\nabla_i \phi$$

where ϕ is the potential. In our discussion, we've found the geodesic equation

$$\frac{D^2 x^\mu}{d\tau^2} = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau} = 0$$

When considering the four-velocity v^μ, we've found

$$v^\mu v_\mu = -c^2 \dot{t} + \dot{x} + \dot{y} + \dot{z} = -c^2$$

So for slow motion, we have the approximation $\dot{x} \ll c\dot{t}$. To make progress, we need to assume that $g_{\mu\nu}$ is static (or the particle's motion could be an artifact of the coordinates). We then find

$$\begin{aligned} \Gamma^\mu{}_{00} = & \frac{1}{2} g^{\mu\nu} \Big(\frac{\partial g_{\nu 0}}{\partial x^0} + \frac{\partial g_{\nu 0}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\nu}\Big) \\ = & -\frac{1}{2} g^{\mu\nu} \frac{\partial g_{00}}{\partial x^\nu} \end{aligned}$$

Then

$$\frac{D^2 t}{d\tau^2} = \frac{d^2 t}{d\tau^2} - \frac{1}{2} g^{t\nu} \frac{\partial g_{00}}{\partial x^\nu} \Big(\frac{dt}{d\tau}\Big)^2 = 0$$

Assuming gravity is weak, we can write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$. The second term drops out, and

$$\frac{Dt}{d\tau} = \text{const.}$$

For the spatial part

$$\frac{d^2x^i}{d\tau^2} - \frac{1}{2}g^{ij}\frac{\partial g_{00}}{\partial x^j}\left(\frac{dt}{d\tau}\right)^2 = 0$$
$$\frac{d^2x^i}{dt^2} - \frac{1}{2}g^{ij}\frac{\partial g_{00}}{\partial x^j} = 0$$

Comparing this to $a = -\nabla\phi$, it seems we want $\partial_i\phi = -\frac{1}{2}\frac{\partial g_{00}}{\partial x^i}$, so $h_{00} = -2\phi$ and

$$g_{00} = -1 + h_{00} = -(1 + 2\phi)$$

This shows us how to relate the Newtonian equations of motion to the geodesics of general relativity in the Newtonian limit.

3 Developing the Einstein equations

Now we want the equivalent of $\nabla^2\phi = 4\pi\rho$, which relates the potential (metric) to matter. Our only clue is the stress tensor conservation $\nabla^\mu T_{\mu\nu} = 0$. We need a tensor $\nabla^\mu X_{\mu\nu} = \kappa\nabla^\mu T_{\mu\nu} = 0$. We have considered $X_{\mu\nu} = G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu}$. This tensor satisfies $\nabla^\mu G_{\mu\nu} = 0$. Taking the trace of both sides,

$$-R = \kappa T,$$

using which we write

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})$$

So we need the stress tensor to calculate gravity.

$$T_{\mu\nu} = m\int \frac{\delta^4(x^\alpha - z^\alpha(\tau))}{\sqrt{-g}}\dot{z}^\mu\dot{z}^\nu d\tau$$

with the action

$$S = -mc\int \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}d\tau d^4x$$

we have

$$T_{\mu\nu} = \dot{z}^\mu\dot{z}^\nu\frac{m\delta^3(x^\mu - z^\mu(t))}{\sqrt{-g}\dot{t}}$$

For dust,

$$T_{\mu\nu} = \rho u_\mu u_\nu,$$
$$T = \rho u^\mu u_\mu = -\rho c^2$$
$$T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu} = \rho(u_\mu u_\nu + \frac{1}{2}g_{\mu\nu}c^2)$$

Using $\nabla^2\phi = 4\pi\rho$ will help us figure out what κ is.