Derivation of the Christoffel symbols directly from the geodesic equation

We start by considering the action for a point particle:

$$S[x^\sigma] = \frac{1}{2}m\int \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}g_{\mu\nu}(x^\sigma)d\lambda.$$

where the orbit is parameterized by λ, and we have stressed that the metric depends on position. Then:

$$\delta S[x^\sigma] = -\frac{1}{2}m\int\left[\frac{d}{d\lambda}\left(g_{\mu\nu}(x^\sigma)\frac{dx^\nu}{d\lambda}\right)\delta x^\mu + \frac{d}{d\lambda}\left(g_{\mu\nu}(x^\sigma)\frac{dx^\mu}{d\lambda}\right)\delta x^\nu - \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\frac{\partial g_{\mu\nu}(x^\sigma)}{\partial x^\rho}\delta x^\rho\right]d\lambda$$

$$= -\frac{1}{2}m\int\left[\frac{d}{d\lambda}\left(g_{\rho\nu}(x^\sigma)\frac{dx^\nu}{d\lambda}\right)\delta x^\rho + \frac{d}{d\lambda}\left(g_{\mu\rho}(x^\sigma)\frac{dx^\mu}{d\lambda}\right)\delta x^\rho - \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\frac{\partial g_{\mu\nu}(x^\sigma)}{\partial x^\rho}\delta x^\rho\right]d\lambda$$

$$= -\frac{1}{2}m\int\left[\frac{d^2x^\nu}{d\lambda^2}g_{\rho\nu}(x^\sigma) + \frac{d^2x^\mu}{d\lambda^2}g_{\mu\rho}(x^\sigma) + \left(\frac{\partial g_{\rho\nu}(x^\sigma)}{\partial x^\mu} + \frac{\partial g_{\mu\rho}(x^\sigma)}{\partial x^\nu} - \frac{\partial g_{\mu\nu}(x^\sigma)}{\partial x^\rho}\right)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right]\delta x^\rho d\lambda.$$

Thus, the equation of motion is:

$$\frac{1}{2}m\left[\frac{d^2x^\nu}{d\lambda^2}g_{\rho\nu}(x^\sigma) + \frac{d^2x^\mu}{d\lambda^2}g_{\mu\rho}(x^\sigma) + \left(\frac{\partial g_{\rho\nu}(x^\sigma)}{\partial x^\mu} + \frac{\partial g_{\mu\rho}(x^\sigma)}{\partial x^\nu} - \frac{\partial g_{\mu\nu}(x^\sigma)}{\partial x^\rho}\right)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right] = 0.$$

Multiplying by $g^{\tau\rho}(x^\sigma)$ and, in the second term, using the fact that $g_{\mu\rho}(x^\sigma) = g_{\rho\mu}(x^\sigma)$, we find:

$$m\left[\frac{d^2x^\tau}{d\lambda^2} + \frac{1}{2}g^{\tau\rho}(x^\sigma)\left(\frac{\partial g_{\rho\nu}(x^\sigma)}{\partial x^\mu} + \frac{\partial g_{\mu\rho}(x^\sigma)}{\partial x^\nu} - \frac{\partial g_{\mu\nu}(x^\sigma)}{\partial x^\rho}\right)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right] = 0.$$

Now, inside the large square brackets, the second term can be identified as:

$$\Gamma^\tau_{\mu\nu}(x^\sigma)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}, \quad \text{since} \quad \Gamma^\tau_{\mu\nu}(x^\sigma) = \frac{1}{2}g^{\tau\rho}(x^\sigma)\left(\frac{\partial g_{\rho\nu}(x^\sigma)}{\partial x^\mu} + \frac{\partial g_{\mu\rho}(x^\sigma)}{\partial x^\nu} - \frac{\partial g_{\mu\nu}(x^\sigma)}{\partial x^\rho}\right),$$

while first term inside the large square brackets can be rewritten as:

$$\frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}\left(\frac{dx^\tau}{d\lambda}\right).$$

Together these can be arranged as:

$$\frac{dx^\mu}{d\lambda}\left[\frac{\partial}{\partial x^\mu}\left(\frac{dx^\tau}{d\lambda}\right) + \Gamma^\tau_{\mu\nu}(x^\sigma)\frac{dx^\nu}{d\lambda}\right] = \frac{dx^\mu}{d\lambda}\nabla_\mu\left(\frac{dx^\tau}{d\lambda}\right),$$

since the two terms in the large square brackets represent the covariant derivative of the vector $dx^\tau/d\lambda$. This variational problem gives the easiest way to find all the $\Gamma^\tau_{\mu\nu}(x^\sigma)$ symbols.