PHZ 6607 Special and General Relativity
August 24, 2018

Nathaniel Strauss (Suzanne Rosenzweig)

1 Electromagnetic Motivation

We start our study of general relativity by drawing an analogy from the example of classical electromagnetism. The important quantities in electromagnetism are the electric and magnetic fields E^i and B^i, which we can calculate from the scalar potential ϕ and vector potential A_i. We can combine the scalar and vector potentials into the four potential A_μ, as well as decompose the vector potential into transverese and longitudinal parts:

$$A_\mu = (\phi, A_i) = (A_0, A_i^T + \partial_i A), \tag{1}$$

where $\nabla \cdot A^T = 0$. In the hamiltonian formulation, the E^i components serve as canonical momenta, which are constrained by the equation $\nabla \cdot E = \rho$. Often, canonical constraints are associated with gauge freedom (as is the case here), but not always.

So what does this have to do with general relativity? Well, both theories involve a Coulomb-like force, so it may seem finding a potential $\widetilde{A}_\mu$ for gravity would be a good place to start. It turns out that this is not possible since there is no negative mass like there is negative charge. Instead of the potential, what we need is called the metric $g_{\mu\nu}$. However, like the potential, the metric is not quite physical, which stems from the fact that physics is coordinate invariant. What is physical is the curvature tensor $R^a{}_{bcd}$, which is analogous to the electromagnetic fields E^i and B_i.

2 An Example: Flat Cartesian Space

All physical phenomena in physics should be coordinate invariant. In order to explicate this fact, we use the example of flat Cartesian space and search for coordinate-invariant quantities. In Fig. 1 are two coordinate systems for flat space, one rotated with respect to the other.

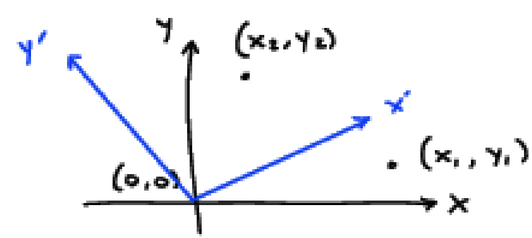

Figure 1: Two coordinate systems for flat space.

Clearly the coordinates for a point is not invariant between the two coordinate systems, but the distance between those points δl is invariant:

$$\delta l^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \tag{2}$$
$$= (x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 \tag{3}$$

Thus, coordinates are not physical, but length is physical.

Particle velocity is another quantity to consider. The velocity itself is not invariant, since

$$v^i = \frac{dx^i}{dt} \neq v^{i'}, \tag{4}$$

where $v^{i'} = dx^{i'}/dt$. However, we will see that the speed, defined something like $|v|^2 = |dx|^2/dt^2 = |dx'|^2/dt^2$ is an invariant quantity. This is because $dl^2 = dx^2 + dy^2 = dx'^2 + dy'^2$ is invariant. We see a lot of invariant information is contained in infinitesimal quantities.

3 Vectors

Vectors are a fundamental object in general relativity, and there are two types. For example, take the velocity of a particle $v^i = dx^i/dt$, where x describes a trajectory of a particle, and $\nabla_i f$, where $f = f(x^i)$ is a scalar function over a three dimensional space, as shown in Fig. 2.

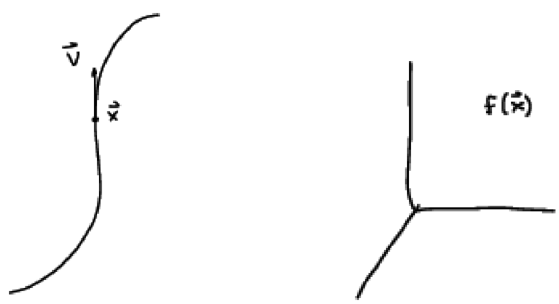

Figure 2: On the left is the path $\vec{x}$ of a particle and its velocity $\vec{v}$. On the right is a scalar function over a three-dimensional space.

We can write

$$v^{i'} = \frac{dx^{i'}}{dt} = \frac{\partial x^{i'}}{\partial x^i} v^i, \tag{5}$$

where $v^{i'}$ are the components of v in the primed system. Contrarily

$$\nabla_i f = \frac{\partial x^i}{\partial x^{i'}} \nabla_i f. \tag{6}$$

(Note: There is ALWAYS one index up and one index down of the same letter for summations.) Notice that the primed coordinate is in the numerator for the v transformation and in the denominator for the ∇f transformation. This means v^i and $\nabla_i f$ are different kinds of vectors, as signified by their index placement.

The coordinate-invariant way to construct the magnitude of vectors is using the metric, which we'll see in the next section. However, we can still construct a coordinate-invariant scalar from the two kinds of vectors we have so far:

$$v^{i'}\nabla_{i'}f = v^i \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{i'}}\nabla_j f = v^i \nabla_i f.$$

4 The Metric

One way to define the metric g_{ij} is by using infinitesimal lengths:

$$ds^2 = dx^2 + dy^2 + dz^2 \equiv g_{ij}dx^i dx^j, \tag{7}$$

so

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

If we make the coordinate transformation

$$\begin{aligned} x &= r\cos\phi\sin\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\theta, \end{aligned} \tag{9}$$

we get

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2, \tag{10}$$

or

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{11}$$

We can now easily write down the velocity of a particle at a point using Eq. 10:

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2.$$

We could do this just as easily for the coordinates in Eq. 7.

For general vectors v^i and w_i, we define their magnitudes

$$\begin{aligned} |v|^2 &= v^i v^j g_{ij} \\ |w|^2 &= w_i w_j g^{ij}, \end{aligned}$$

where

$$g^{ij}g_{jk} = \delta^i{}_k.$$

Finally, we can use this to construct $u_i = v^j g_{ji}$. Then $u_i v^i = |v|^2$. For this reason, we define for every vector v^i, $v_i = v^j g_{ji}$, so $|v|^2 = v_i v^i$. Similarly, we define for every vector w_i, $w^i = w_j g^{ji}$.