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1. Coordinates for Schwarzchild and Maximal Extension

When looking at light so far we have been looking at light rays, when talking about light waves or other fields we use the equation

$$-\partial_t^2 + \partial_{r_*}^2 - (1 - \frac{2M}{r})(\frac{l(l+1)}{r^2} - \frac{2m(1-s^2)}{r^3}) \tag{1}$$

The angular dependence of the entire field is given by the spherical harmonics Y_{lm}, though this dependence is not in Eq 1. s is equal to 0, 1, or 2 for scalar fields, electromagnetism, and gravity respectively. We see that for scalars and gravity there is a higher order $\frac{1}{r^4}$ term.

$$r^* = r + 2M\ln(\frac{r}{2M} - 1) \tag{2}$$

We see that as $r \to \infty, r_* \to \infty$, and $r \to 2M, r_* \to -\infty$. We also have

$$dr_* = \frac{dr}{1 - \frac{2M}{r}} \tag{3}$$

The metric is then

$$ds^2 = (1 - \frac{2M}{r})(-dt^2 + dr_*^2) + r^2 d\Omega^2 \tag{4}$$

For null rays this results in

$$dt = \pm\frac{dr}{1 - \frac{2M}{r}} = dr_* \tag{5}$$

Let $u = t + r_*$ and $v = t - r_*$. Consider using the coordinates u and r.
The new metric is

$$ds^2 = (1 - \frac{2M}{r})dv^2 + 2dvdr + r^2 d\Omega^2 \tag{6}$$

The determinant is $g = -r^4\sin^2(\theta)$, and is fine at $r = 2M$.
For null curves we have $\frac{dv}{dr}$ is zero when ingoing, and when outgoing it is

$$\frac{dv}{dr} = 2(1 - \frac{2M}{r})^{-1} \tag{7}$$

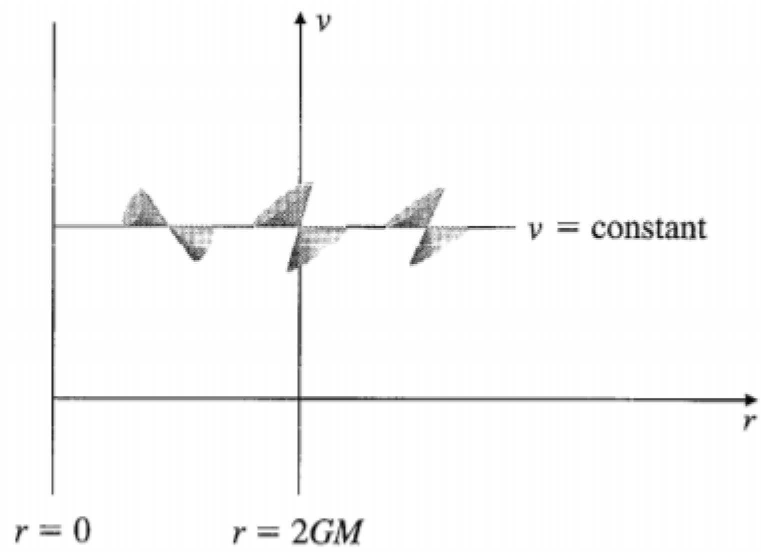

Figure 1. tipping of light cones in v and r coordinates

Nothing Inside $r = 2M$ can go back. We can use u instead of v to get the following metric

$$ds^2 = (1 - \frac{2M}{r})du^2 - 2dvdr + r^2 d\Omega^2 \tag{8}$$

Now we have for outgoing light rays that $\frac{du}{dr} = 0$ and for incoming light rays

$$\frac{du}{dr} = -2(1 - \frac{2M}{r})^{-1} \tag{9}$$

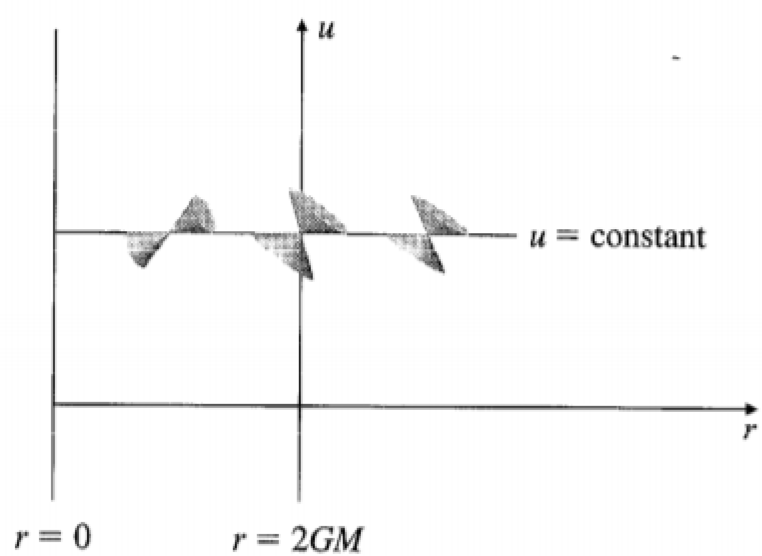

Figure 2. tipping of light cones in u and r coordinates

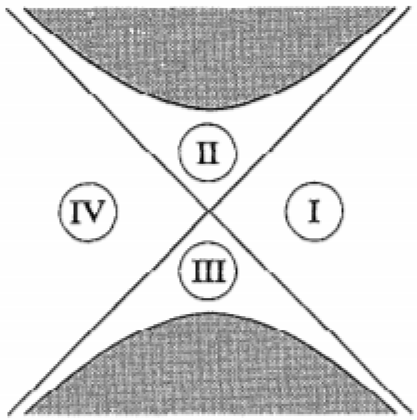

Figure 3. Four Regions of Maximal Extension of Schwarzchild

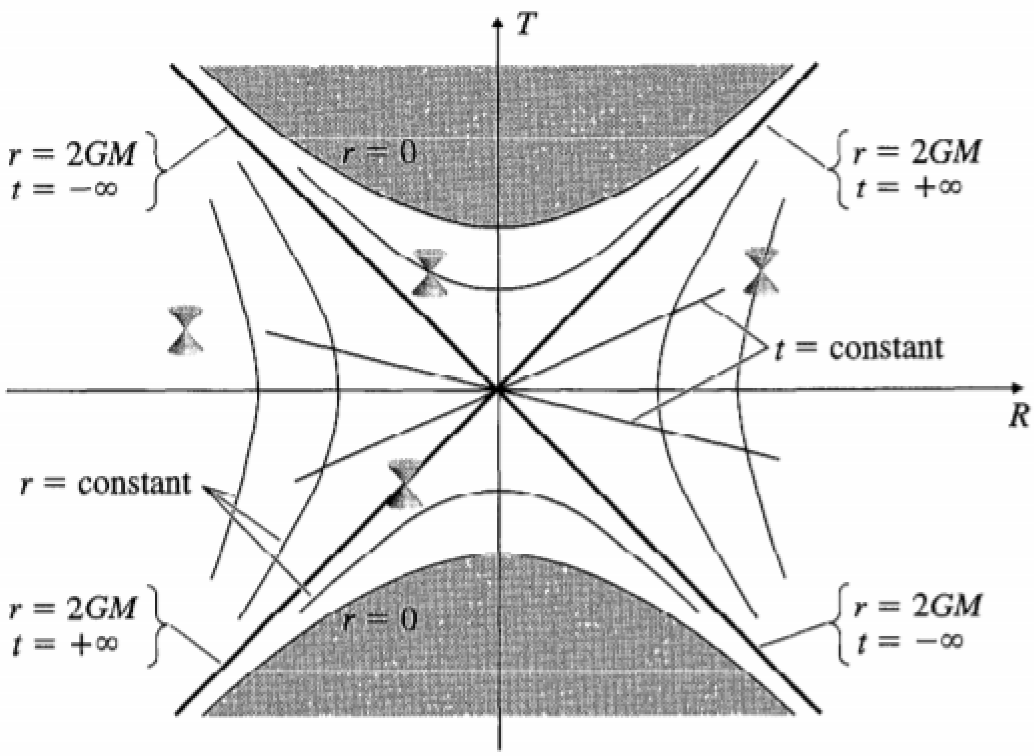

Figure 4. Maximal Extension Light Cones and Labeled Curves

Now nothing can go back in $r = 2M$.

We can use the coordinates $v' = e^{\frac{v}{4m}}$ and $u' = -e^{\frac{-u}{4m}}$ We can then define the coordiantes $T = \frac{1}{2}(u' + v')$ and $R = \frac{1}{2}(v' - u')$. We then have

$$T^2 - R^2 = v'u' = e^{\frac{r}{2m}}(1 - \frac{r}{2m}) \tag{10}$$

Note that the product $u'v'$ smoothly changes sign as r crosses $2M$.

With the coordinates u' and v', their product must be less than one, hence the shaded regions in Figure 3 are excluded. There are finite range coordinates with

$u'' = \arctan(u')$ and $v'' = \arctan(v')$. Regions 1 and 2 are what we typically think of as a black hole, while regions 3 and 4 are called a white hole. We can only relate t and r to T and R in the above way in the first region. The four regions are the maximal extension of the Schwarzchild geometry. This still is not a collapsing star.