PHY 6646 - Quantum Mechanics II - Spring 2020
Homework set # 12, due April 8

1. Consider a particle of mass m moving in a potential $V(x) = -\frac{1}{2}\kappa x^2$, sometimes called the "Inverted Harmonic Oscillator".

i. Solve the classical equation of motion of such a particle for arbitrary initial conditions $x(0)$ and $\dot{x}(0) \equiv \frac{dx}{dt}(0)$, at $t = 0$. Does the particle oscillate?

ii. What is the classical Lagrangian for this system? What variable p is canonically conjugate to x? What is the classical Hamiltonian? Write Hamilton's equations of motion for $x(t)$ and $p(t)$.

iii. Write down the Hamiltonian for the quantum system. Define

$$a = \frac{1}{\sqrt{2}}\left(\frac{X}{\Delta} + i\frac{\Delta}{\hbar}P\right) \tag{0.1}$$

where $\Delta = \sqrt{\frac{\hbar}{m\gamma}}$ with $\gamma = \sqrt{\frac{\kappa}{m}}$. What meaning does γ have with regard to the classical motion? Show that $[a, a^\dagger] = 1$, and

$$H = -\frac{1}{2}\gamma\hbar(aa + a^\dagger a^\dagger) \quad . \tag{0.2}$$

Let

$$|n\rangle \equiv \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \tag{0.3}$$

where the state $|0\rangle$ is defined by

$$a|0\rangle = 0 \quad . \tag{0.4}$$

Show that

$$\psi_0(x) \equiv \langle x|0\rangle = \frac{1}{\sqrt{\sqrt{\pi}\Delta}}e^{-\frac{1}{2}(\frac{x}{\Delta})^2} \quad . \tag{0.5}$$

Do the $|n\rangle$ for $n = 0, 1, 2, 3...$ form a complete orthonormal set of statevectors? Why?

iv. In the Heisenberg picture, what is the equation of motion for $a(t)$? Show that it is solved by

$$a(t) = \cosh(\gamma t)a(0) + i\sinh(\gamma t)a^\dagger(0) \quad . \tag{0.6}$$

Consider the Heisenberg state $|\psi\rangle = |0\rangle$, and the operator $N(t) \equiv a^\dagger(t)a(t)$. Evaluate $\langle\psi|N(t)|\psi\rangle$, $\langle\psi|N(t)^2|\psi\rangle$ and the root mean square deviation $\delta N(t) = \sqrt{\langle N(t)^2\rangle - \langle N(t)\rangle^2}$ of $N(t)$ in that state.

v. In the Schrödinger picture, consider the state vector

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n(t)|n\rangle \tag{0.7}$$

where $|\psi(0)\rangle = |0\rangle$. Show that

$$a(-t)|\psi(t)\rangle = 0 \tag{0.8}$$

where $a(t)$ is the Heisenberg operator of part iv. Using Eq.(0.8), obtain a recursion relation between the coefficients $c_n(t)$. Show that the recursion relation implies that $c_n(t) = 0$ for all n odd, and

$$c_{2p}(t) = (i\tanh(\gamma t))^p \sqrt{\frac{(2p-1)!!}{2^p p!}}\, c_0(t) \tag{0.9}$$

for $n = 2p$. Show that the normalization condition $\langle\psi(t)|\psi(t)\rangle = 1$ implies

$$c_0(t) = \frac{1}{\sqrt{\cosh(\gamma t)}} \quad . \tag{0.10}$$

What is the probability that the system is in state $|n\rangle$ at time t. What is the average outcome of measuring $N = a^\dagger a$ at time t? Is the result the same as in part iv?

2. Problem 21.2.2 in Shankar's book.