PHY 6645 - Quantum Mechanics I - Fall 2018
Homework #8, due October 17

1. Consider a particle of mass μ moving in one dimension in a potential $V(x)$:

$$\begin{aligned} V(x) &= 0 \qquad \text{for } x < -a \\ &= V_\infty \qquad \text{for } x > a \quad , \end{aligned} \tag{0.1}$$

where V_∞ is a constant. In the range $-a < x < a$, $V(x)$ takes arbitrary values. The wavefunction describing a particle incident from the left with energy $E = \frac{\hbar^2 k^2}{2\mu} = \frac{\hbar^2 k'^{\,2}}{2\mu} + V_\infty$ has the properties

$$\begin{aligned} \psi_E(x) &= e^{ikx} + A_-(k)e^{-ikx} \qquad \text{for } x < -a \\ &= B(k)e^{ik'x} \qquad \text{for } x > a \quad . \end{aligned} \tag{0.2}$$

a. Show that for any momentum space wavefunction $\tilde{\psi}_I(p)$, the time-dependent Schrödinger equation is solved by

$$\psi(x,t) = \int_{-\infty}^{+\infty} \frac{dp}{\sqrt{2\pi\hbar}} \; \tilde{\psi}_I(p) \; \psi_E(x) e^{-\frac{i}{\hbar}Et} \tag{0.3}$$

where $p = \hbar k$ and $E = \frac{p^2}{2\mu}$.

b. Let $\tilde{\psi}_I(t)$ be the momentum space wavefunction for the physical space wavefunction

$$\psi_I(x) = \frac{1}{\sqrt{\sqrt{\pi}\Delta}} e^{ik_0(x-x_0) - \frac{1}{2\Delta^2}(x-x_0)^2} \quad , \tag{0.4}$$

describing a Gaussian wavepacket localized at x_0 with average momentum $p_0 = \hbar k_0$. Assume that $x_0 < 0$ with $|x_0| >> a$ and $p_0 > 0$. Show that for $x < -a$ and $t = 0$

$$\psi(x,0) \simeq \psi_I(x) \tag{0.5}$$

where $\simeq$ means equal up to terms that are exponentially small in the limit $\Delta << |x_0|$, $\Delta p = \frac{\hbar}{\sqrt{2}\Delta} << p_0$ and $\Delta p << p'_0$.

c. Show that for $x > a$ and $t = 0$

$$\psi(x,0) \simeq B(k_0) e^{i(k'_0 - \frac{k_0^2}{k'_0})x} \psi_I(\frac{k_0}{k'_0}x) \quad . \tag{0.6}$$

Show that $\psi(x,0) \simeq 0$ for $x > a$.

d. Show that for $t > 0$ and $x < -a$

$$\psi(x,t) = \psi_{\rm inc}(x,t) + \psi_{\rm refl}(x,t) \tag{0.7}$$

where

$$\psi_{\rm inc}(x,t) = \psi_0(x,t) \equiv \frac{1}{\sqrt{\sqrt{\pi}(\Delta + \frac{i\hbar t}{\mu\Delta})}} e^{-\frac{(x-x_0-p_0t/\mu)^2}{2\Delta^2(1+i\hbar t/\mu\Delta^2)} + \frac{i}{\hbar}p_0(x-x_0) - \frac{i}{\hbar}\frac{p_0^2}{2\mu}t} \tag{0.8}$$

is the time evolution of $\psi_I(x) = \psi_0(x,0)$ in empty space (i.e. in the absence of potential) and

$$\psi_{\rm refl}(x,t) \simeq A_-(k_0)\psi_0(-x,t) \quad . \tag{0.9}$$

Show that $\psi_{\rm inc}(x,t)$ describes a wavepacket moving to the right with velocity $v = \frac{p_0}{\mu}$ and disappearing from the $x < -a$ region after a time of order $t_0 = \frac{|x_0|}{v}$. Also show that $\psi_{\rm refl}(x,t)$ describes a wavepacket that appears at a time of order t_0 near $x = 0$ and travels to the left with velocity $-v$.

e. Show that for $t > 0$ and $x > a$

$$\psi(x,t) = \psi_{\rm trans}(x,t) \simeq B(k_0) e^{i(k_0' - \frac{k_0^2}{k_0'})x} \psi_0(\frac{k_0}{k_0'} x, t) \quad . \tag{0.10}$$

Show that $\psi_{\rm trans}(x,t)$ describes a wavepacket appearing near $x = 0$ at a time of order t_0 and traveling to the right with velocity $v' = \frac{p_0'}{\mu}$.

f. Show that for any wavepacket

$$\psi(x,t) = \int_{-\infty}^{+\infty} dp \frac{1}{\sqrt{2\pi\hbar}} \tilde{\psi}(p) e^{\frac{i}{\hbar}(p' x - \frac{p^2}{2\mu} t)} \tag{0.11}$$

with $\frac{p^2}{2\mu} = E = \frac{p'^2}{2\mu} + V_\infty$, the time integral of the probability current

$$\text{Prob.} = \int_{-\infty}^{+\infty} dt \; j(x,t) \simeq \frac{p_0'}{p_0} \int_{-\infty}^{+\infty} dp \; |\tilde{\psi}(p)|^2 \tag{0.12}$$

if $\tilde{\psi}(p)$ is strongly peaked near p_0, so that the momentum spread $\Delta p << p_0'$. Show that Prob. $= 1$ and $R(k_0)$ respectively for the incoming and reflected wavepackets found in part d., and Prob. $= T(k_0)$ for the transmitted wavepacket found in part e., where $R(k)$ and $T(k)$ are the reflection and transmission coefficients.

2. Problem 7.3.4 in Shankar's book.