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R.P. Woodard

Solutions to Asmt #9

(1a) * $\dot{Q}_1 = \partial H/\partial P_1 = Q_2$ ✓

(1b) * $\dot{Q}_2 = \frac{\partial H}{\partial P_2} = a + \cancel{P_2 \frac{\partial a}{\partial P_2}} - \cancel{\overbrace{\frac{\partial L}{\partial \dot{q}}}^{P_2} \frac{\partial a}{\partial P_2}} = a$ ✓

(1c) * $\dot{P}_2 = -\frac{\partial H}{\partial Q_2} = -P_1 - \cancel{P_2 \frac{\partial a}{\partial Q_2}} + \frac{\partial L}{\partial \dot{q}} + \cancel{\frac{\partial L}{\partial \dot{q}} \frac{\partial a}{\partial Q_2}} \Rightarrow P_1 = \frac{\partial L}{\partial \dot{q}} - \dot{P}_2 = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)$ ✓

(1d) * $\dot{P}_1 = -\frac{\partial H}{\partial Q_1} = -\cancel{P_2 \frac{\partial a}{\partial Q_1}} + \frac{\partial L}{\partial q} + \cancel{\frac{\partial L}{\partial \ddot{q}} \frac{\partial a}{\partial Q_1}} \Rightarrow \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)\right] = \frac{\partial L}{\partial q}$ ✓

(1e) * $\dot{H} = \dot{P}_1 Q_2 + P_1 \dot{Q}_2 + \dot{P}_2 a + \cancel{P_2 \dot{a}} - \frac{\partial L}{\partial q} \dot{Q}_1 - \frac{\partial L}{\partial \dot{q}} \dot{Q}_2 - \cancel{\frac{\partial L}{\partial \ddot{q}} \dot{a}}$

$= \frac{\partial L}{\partial q} Q_2 + P_1 a + \left(\frac{\partial L}{\partial \dot{q}} - P_1\right) a - \frac{\partial L}{\partial q} Q_2 - \frac{\partial L}{\partial \dot{q}} a = 0$ ✓

(2a) * $P \equiv \frac{\partial L}{\partial \dot{q}} = e^{\gamma t} m \dot{q} \Rightarrow \dot{q} = e^{-\gamma t} P/m$

* $H \equiv P\dot{q} - L = e^{-\gamma t} \frac{P^2}{2m} + e^{+\gamma t} \frac{1}{2} k Q^2$

(2b) * $\left\{ \begin{array}{l} \dot{Q} = \partial H/\partial P = e^{-\gamma t} P/m \\ \dot{P} = -\frac{\partial H}{\partial Q} = -e^{\gamma t} k Q \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \ddot{Q} = -\gamma \dot{Q} - \frac{k}{m} Q \\ P = e^{\gamma t} m \dot{Q} \end{array} \right\}$

NB this is the same as the Euler-Lagrange eqn
Hamiltonian techniques do not make it any easier to solve

* From problem 3 of Asmt #1

$$Q(t) = e^{-\frac{\gamma}{2}t}\left[Q_0 \cos(\omega' t) + \frac{1}{\omega'}\left(\frac{P_0}{m} + \frac{1}{2}\gamma Q_0\right)\sin(\omega' t)\right]$$
$$P(t) = e^{+\frac{\gamma}{2}t}\left[P_0 \cos(\omega' t) - \frac{1}{\omega'}\left(\frac{1}{2}\gamma P_0 + k Q_0\right)\sin(\omega' t)\right]$$

where $\omega'^2 = \frac{k}{m} - \frac{1}{4}\gamma^2$

(2c) * $H(t) = \frac{1}{2}\left(\frac{P_0^2}{m} + kQ_0^2\right)\cos^2(\omega' t) + \frac{\gamma}{2\omega'}\left(kQ_0^2 - \frac{P_0^2}{m}\right)\sin(\omega' t)\cos(\omega' t) + \frac{1}{2\omega'^2}\left[\left(\frac{k}{m} + \frac{1}{4}\gamma^2\right)\frac{P_0^2}{m} + \frac{2\gamma k}{m} Q_0 P_0 + \left(\frac{k}{m} + \frac{1}{4}\gamma^2\right) k Q_0^2\right]\sin^2(\omega' t)$

(3a) * $P_i \equiv \frac{\partial L}{\partial \dot{r}_i} = \frac{mc\dot{r}_i}{\sqrt{c^2 - \|\dot{\vec{r}}\|^2}} + qA_i$ ie $\vec{P} = \frac{mc\dot{\vec{r}}}{\sqrt{c^2 - \|\dot{\vec{r}}\|^2}} + q\vec{A}$

* inverting gives $\dot{\vec{r}} = c[\vec{P} - q\vec{A}]/\sqrt{m^2c^2 + \|\vec{P} - q\vec{A}\|^2}$

(3b) * NB $\sqrt{c^2 - v^2} = \frac{mc^2}{\sqrt{m^2c^2 + \|\vec{P} - q\vec{A}\|^2}}$

* $H = \vec{P} \cdot \dot{\vec{r}} - L = \sqrt{m^2c^4 + c^2\|\vec{P} - q\vec{A}\|^2} + qV$

* NB minimal coupling $\left\{ \begin{array}{l} \vec{P} \rightarrow \vec{P} - q\vec{A} \\ H \rightarrow H + qV \end{array} \right\}$