Solutions to Asmt #5

① * $(A^T \dot{A})_{ij} = A_{ki} \dot{A}_{kj}$

* In a problem like this it is best to label each term in the two factors & then work out each product

* $A_{ki} = \overset{①}{[\delta_{ki} - \hat{n}_k \hat{n}_i]} \cos(\Phi) + \overset{②}{\epsilon_{ki\ell} \hat{n}_\ell \sin(\Phi)} + \overset{③}{\hat{n}_k \hat{n}_i}$

* $\dot{A}_{kj} = -\overset{ⓐ}{[\delta_{kj} - \hat{n}_k \hat{n}_j] \sin(\Phi) \dot{\Phi}} + \overset{ⓑ}{\epsilon_{kj\ell} \hat{n}_\ell \cos(\Phi) \dot{\Phi}} + \overset{ⓒ}{[\dot{\hat{n}}_k \hat{n}_j + \hat{n}_k \dot{\hat{n}}_j][1 - \cos(\Phi)]} + \overset{ⓓ}{\epsilon_{kjm} \dot{\hat{n}}_m \sin(\Phi)}$

$(1a) = ① * ⓐ = -[\delta_{ij} - \hat{n}_i \hat{n}_j] \sin(\Phi) \cos(\Phi) \dot{\Phi}$

$(2a) = \epsilon_{ijk} \hat{n}_k \sin^2(\Phi) \dot{\Phi}$

$(3a) = 0$

$(1b) = \epsilon_{ijk} \hat{n}_k \cos^2(\Phi) \dot{\Phi}$

$(2b) = +[\delta_{ij} - \hat{n}_i \hat{n}_j] \sin(\Phi) \cos(\Phi) \dot{\Phi}$

$(3b) = 0$

$\Big\} = \epsilon_{ijk} \hat{n}_k \dot{\Phi}$

$(1c) = \hat{n}_i \dot{\hat{n}}_j [\cos(\Phi) - \cos^2(\Phi)]$

$(2c) = \epsilon_{ik\ell} \dot{\hat{n}}_k \hat{n}_\ell \hat{n}_j [\sin(\Phi) - \sin(\Phi) \cos(\Phi)]$

$(3c) = \hat{n}_i \dot{\hat{n}}_j [1 - \cos(\Phi)]$

$(1d) = [\epsilon_{ijk} \dot{\hat{n}}_k + \hat{n}_i \epsilon_{jk\ell} \hat{n}_k \dot{\hat{n}}_\ell] \sin(\Phi) \cos(\Phi)$

$(2d) = -\dot{\hat{n}}_i \hat{n}_j \sin^2(\Phi)$

$(3d) = -\hat{n}_i \epsilon_{jk\ell} \hat{n}_k \dot{\hat{n}}_\ell \sin(\Phi)$

* $(1c) + (2d) = -\dot{\hat{n}}_i \hat{n}_j [1 - \cos(\Phi)]$
* $(3c) = +\hat{n}_i \dot{\hat{n}}_j [1 - \cos(\Phi)]$

$\Big\} = \epsilon_{ijk} \epsilon_{k\ell m} \hat{n}_\ell \dot{\hat{n}}_m [1 - \cos(\Phi)]$

* $(1d) + (3d) = \epsilon_{ijk} \dot{\hat{n}}_k \sin(\Phi) \cos(\Phi) - \hat{n}_i \epsilon_{jk\ell} \hat{n}_k \dot{\hat{n}}_\ell [\sin(\Phi) - \sin(\Phi) \cos(\Phi)]$

* $(2c) = +\hat{n}_j \epsilon_{ik\ell} \dot{\hat{n}}_k \hat{n}_\ell [\sin(\Phi) - \sin(\Phi) \cos(\Phi)]$

→ $(1d) + (3d) + (2c) = \epsilon_{ijk} \dot{\hat{n}}_k \sin(\Phi) \cos(\Phi) - \epsilon_{ijk} \epsilon_{k\ell m} \hat{n}_\ell \epsilon_{mnp} \hat{n}_n \dot{\hat{n}}_p [\sin(\Phi) - \sin(\Phi) \cos(\Phi)]$

$= \epsilon_{ijk} \dot{\hat{n}}_k \sin(\Phi) \cos(\Phi) - \epsilon_{ijk} [\delta_{kn} \delta_{\ell p} - \delta_{kp} \delta_{\ell n}] \hat{n}_\ell \hat{n}_n \dot{\hat{n}}_p [\sin(\Phi) - \sin(\Phi) \cos(\Phi)]$

$\hat{n}_k \hat{n} \cdot \dot{\hat{n}} \to 0 \quad - \dot{\hat{n}}_k \hat{n} \cdot \hat{n} = -\dot{\hat{n}}_k$

$= \epsilon_{ijk} \dot{\hat{n}}_k \sin(\Phi)$

∴ $(A^T \dot{A})_{ij} = \epsilon_{ijk} \Big[\hat{n}_k \dot{\Phi} + \dot{\hat{n}}_k \sin(\Phi) + \epsilon_{k\ell m} \hat{n}_\ell \dot{\hat{n}}_m [1 - \cos(\Phi)] \Big]$

* NB $\dot{\hat{n}}$ & $\hat{n} \times \dot{\hat{n}}$ are the two independent vectors ⊥ $\hat{n}$

* NB $\vec{\omega} = \hat{n} \dot{\Phi} + \dot{\hat{n}} \sin(\Phi) + \hat{n} \times \dot{\hat{n}} [1 - \cos(\Phi)]$ is the angular velocity for the general case in which $\hat{n}$ varies in addition to $\Phi(t)$

Solutions to Asmt #5

(2a) * $V_c = -\frac{1}{2} m (\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r}) \rightarrow -\frac{\partial V_c}{\partial r_i} = m \frac{\partial}{\partial r_i} (\vec{\omega} \times \vec{r})_j * (\vec{\omega} \times \vec{r})_j$

* $\frac{\partial}{\partial r_i} (\vec{\omega} \times \vec{r})_j = \epsilon_{jk\ell} \omega_k \delta_{i\ell} = \epsilon_{ijk} \omega_k$

$\rightarrow -\frac{\partial V_c}{\partial r_i} = m \epsilon_{ijk} \omega_k (\vec{\omega} \times \vec{r})_j = [-m \vec{\omega} \times (\vec{\omega} \times \vec{r})]_i$

(2b) * $\vec{\omega} = \omega \hat{z} \rightarrow V_c = -\frac{1}{2} m \omega^2 r^2 \sin^2(\theta)$ & $V_g = \frac{-m M_E G}{r} \left[1 - J_2 \left[\frac{3}{2}\cos^2(\theta) - \frac{1}{2}\right] \left(\frac{R_E}{r}\right)^2\right]$

* $V_c + V_g = \frac{-m M_E G}{r} \left\{1 - J_2 \left[\frac{3}{2}\cos^2(\theta) - \frac{1}{2}\right] \left(\frac{R_E}{r}\right)^2 + \frac{1}{2} \frac{\omega^2 r^3}{M_E G} \sin^2(\theta)\right\} = \frac{-m M_E G}{R_E} \left\{1 + \frac{1}{2} J_2 + \frac{\omega^2 R_E^3}{2 M_E G}\right.$

$\rightarrow$ $R(\theta) = R_E \left[1 - \frac{3}{2} J_2 \cos^2(\theta) - \frac{1}{2} \frac{\omega^2 R_E^3}{M_E G} \cos^2(\theta) + \text{2nd order}\right]$

* $\Delta R(\theta) \equiv R_E - R(\theta) = R_E \left[\frac{3}{2} J_2 + \frac{1}{2} \frac{\omega^2 R_E^3}{M_E G}\right] \cos^2(\theta) \xrightarrow[\text{largest } \theta = 0]{} R_E \frac{1}{2} \left[3 J_2 + \frac{\omega^2 R_E^3}{M_E G}\right]$

* $T = 8.64 \times 10^4 s$
* $R_E \doteq 6.378 \times 10^6 m$ $\Big\} \rightarrow a_c = R_E \left(\frac{2\pi}{T}\right)^2 \doteq 3.37 \times 10^{-2} m/s^2 \rightarrow \frac{\omega^2 R_E^3}{G M_E} = \frac{a_c}{g} \doteq 0.00344$

$\therefore \Delta R = \frac{1}{2} R_E \left(3 J_2 + \frac{a_c}{g}\right) \doteq 2.13 \times 10^4 m$

(2c) * downward component of force is $\hat{r} \cdot \vec{\nabla} V = \frac{\partial V}{\partial r}$

* downward acceleration $= \frac{G M_E}{r^2} \left\{1 - 3 J_2 \left[\frac{3}{2}\cos^2(\theta) - \frac{1}{2}\right] \left(\frac{R_E}{r}\right)^2 - \frac{r^3 \omega^2}{G M_E} \sin^2(\theta)\right\}$

* evaluating at $r = R(\theta)$ to 1st order $\rightarrow$ $a(\theta) = g \left\{1 + \frac{3}{2} J_2 \sin^2(\theta) + \frac{a_c}{g}\left[1 - 2\sin^2(\theta)\right]\right\}$

~~* max variation is from $\theta = 0$ to $\theta = \frac{\pi}{2}$~~

$\rightarrow \Delta a = g \left(2 \frac{a_c}{g} - \frac{3}{2} J_2\right) \doteq 0.0053 g \doteq 5.2 \times 10^{-2} m/s^2$

(3a) * mass of cylinder $= \pi r^2 \rho_0 * b$

* mass of cone $= \int_0^a dz \, \pi \left(\frac{rz}{a}\right)^2 \rho_0 = \pi r^2 \rho_0 * \frac{1}{3} a$ $\Big\} \rightarrow$ $M = \pi \left(\frac{1}{3} a + b\right) r^2 \rho_0$

(3b) * COM of cylinder $= (a + \frac{1}{2} b) \hat{z}$

* COM of cone $= \int_0^a dz \, z \cdot \left(\frac{rz}{a}\right)^2 \hat{z} \div \int_0^a dz \left(\frac{zr}{a}\right)^2 = \frac{3}{4} a \hat{z}$ $\Big\} \rightarrow \vec{R} = \frac{b \cdot (a + \frac{b}{2}) \hat{z} + \frac{1}{3} a \cdot \frac{3}{4} a \hat{z}}{b + \frac{1}{3} a} = \frac{(\frac{1}{2} b^2 + ab + \frac{1}{4} a^2) \hat{z}}{b + \frac{1}{3} a}$

(3c) * By symmetry off diagonal elements vanish & $I_{11} = I_{22}$

* In cylindrical coords radius$^2 = \rho^2 + z^2 \rightarrow I_{11} = I_{22} = \int d^3 r \rho_0 \left(\frac{\rho^2}{2} + z^2\right)$ & $I_{33} = \int d^3 r \rho_0 \rho^2$

* Cylinder $\begin{cases} I_{11} = I_{22} = 2\pi \rho_0 \int_a^{a+b} dz \int_0^r d\rho \, \rho \left(\frac{\rho^2}{2} + z^2\right) = \pi \rho_0 b r^2 \left[\frac{1}{4} r^2 + a^2 + ab + \frac{1}{3} b^2\right] \\ I_{33} = 2\pi \rho_0 \int_a^{a+b} dz \int_0^r d\rho \, \rho \cdot \rho^2 = \frac{\pi}{2} \rho_0 b r^2 * r^2 \end{cases}$

* Cone $\begin{cases} I_{11} = I_{22} = 2\pi \rho_0 \int_0^a dz \int_0^{zr/a} d\rho \, \rho \left(\frac{\rho^2}{2} + z^2\right) = \frac{\pi}{10} \rho_0 a r^2 \left[\frac{1}{2} r^2 a^2\right] \\ I_{33} = 2\pi \rho_0 \int_0^a dz \int_0^{zr/a} d\rho \, \rho \cdot \rho^2 = \frac{\pi}{10} \rho_0 a r^2 * r^2 \end{cases}$

$\therefore I_{11} = I_{22} = \pi \rho_0 r^2 \left[\frac{1}{4} b r^2 + b a^2 + a b^2 + \frac{1}{3} b^3 + \frac{1}{20} a r^2 + \frac{1}{5} a^3\right]$ & $I_3 = \frac{\pi}{2} \rho_0 r^4 \left(b + \frac{1}{5} a\right)$

(3d) * the eigen values are I_{33} (with eigen vector $\hat{z}$)

& $I_{11} = I_{22}$ (with eigen vectors $\hat{x}$ & $\hat{y}$)