1/7/18
R.P. Woodard

Solutions to Asmt #1

(1a) * In cylindrical coordinates the constraint is $z = k\rho^2 \rightarrow \dot{z} = 2k\rho\dot{\rho}$

* $L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - mgz = \boxed{\frac{1}{2}m(1+4k^2\rho^2)\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 - mgk\rho^2}$

(1b) * $m\rho^2\dot{\phi} = L \rightarrow \dot{\phi} = L/m\rho^2$

* $E = \frac{1}{2}m(1+4k^2\rho^2)\dot{\rho}^2 + \frac{L^2}{2m\rho^2} + mgk\rho^2 \rightarrow \dot{\rho} = \pm\sqrt{\frac{\frac{2E}{m} - \frac{L^2}{m^2\rho^2} - 2gk\rho^2}{1+4k^2\rho^2}}$

(1c) * For a circular orbit (radius R) we need both $\dot{\rho} = 0$ & $\ddot{\rho} = 0$

* $\dot{\rho} = 0 \rightarrow \frac{2E}{m} - \frac{L^2}{m^2R^2} - 2gkR^2 = 0$

* $\ddot{\rho} = 0 \rightarrow \frac{2L^2}{m^2R^3} - 4gkR = 0$

$\rightarrow \left\{ E = \sqrt{2gk}\,L,\quad R = \left(\frac{L^2}{2m^2gk}\right)^{1/4} \right\}$

(1d) * Just expand the force eqn

$(1+4k^2\rho^2)\ddot{\rho} + 4k^2\rho\dot{\rho}^2 = \frac{L^2}{m^2\rho^3} - 2gk\rho$ to $\Delta\rho \equiv \rho - R$

$\rightarrow (1+4k^2R^2)\Delta\ddot{\rho} + 0 = -\left[\frac{3L^2}{m^2R^4} + 2gk\right]\Delta\rho = -8gk\,\Delta\rho$

* the ang. frequency is $\omega^2 = \frac{8gk}{1+4k^2R^2} = \left(\frac{2\pi}{T}\right)^2 \Rightarrow \boxed{T = 2\pi\sqrt{\frac{1+4k^2R^2}{8gk}}}$

(2a) $(\delta R^2)_{ij} = (\delta R)_{ik}(\delta R)_{kj} = \Phi^2\epsilon_{ik\ell}\hat{n}_\ell\epsilon_{kjm}\hat{n}_m = -\Phi^2(\delta_{ij}\delta_{\ell m} - \delta_{im}\delta_{j\ell})\hat{n}_\ell\hat{n}_m$

$= -\Phi^2(\delta_{ij} - \hat{n}_i\hat{n}_j)$

(2b) $(\delta R^3)_{ij} = (\delta R^2)_{ik}(\delta R)_{kj} = -\Phi^2(\delta_{ik} - \hat{n}_i\hat{n}_k) \times \Phi\epsilon_{kj\ell}\hat{n}_\ell = -\Phi^3\epsilon_{ijk}\hat{n}_k$

→ 0 by anti-symmetry of $\epsilon_{kj\ell}$

(2c) * $(\delta R^{2n})_{ij} = \left([-\Phi^2(I - \hat{n}\hat{n})]^n\right)_{ij} = (-\Phi^2)^n(\delta_{ij} - \hat{n}_i\hat{n}_j)$

* $(\delta R^{2n+1})_{ij} = (-1)^n\Phi^{2n+1}\epsilon_{ijk}\hat{n}_k$

(2d) * $\exp[\delta R] = I + \sum_{n=1}^{\infty}\frac{1}{2n!}\delta R^{2n} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}(\delta R)^{2n+1}$

$= \hat{n}_i\hat{n}_j + \sum_{n=0}^{\infty}\frac{(-1)^n}{2n!}\Phi^{2n}[I_{ij} - \hat{n}_i\hat{n}_j] + \sum_{n=0}^{\infty}\frac{(-1)^n\Phi^{2n+1}}{(2n+1)!}\epsilon_{ijk}\hat{n}_k$

$= \hat{n}_i\hat{n}_j + \cos(\Phi)[\delta_{ij} - \hat{n}_i\hat{n}_j] + \sin(\Phi)\epsilon_{ijk}\hat{n}_k$

(2e)

* $\hat{x} \rightarrow \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0\\0&\cos\Phi&0\\0&0&\cos\Phi\end{pmatrix} + \begin{pmatrix}0&0&0\\0&0&\sin\Phi\\0&-\sin\Phi&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&\cos\Phi&\sin\Phi\\0&-\sin\Phi&\cos\Phi\end{pmatrix}$

* $\hat{y} \rightarrow \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} + \begin{pmatrix}\cos\Phi&0&0\\0&0&0\\0&0&\cos\Phi\end{pmatrix} + \begin{pmatrix}0&0&-\sin\Phi\\0&0&0\\+\sin\Phi&0&0\end{pmatrix} = \begin{pmatrix}\cos\Phi&0&-\sin\Phi\\0&1&0\\\sin\Phi&0&\cos\Phi\end{pmatrix}$

* $\hat{z} \rightarrow \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} + \begin{pmatrix}\cos\Phi&0&0\\0&\cos\Phi&0\\0&0&0\end{pmatrix} + \begin{pmatrix}0&\sin\Phi&0\\-\sin\Phi&0&0\\0&0&0\end{pmatrix} = \begin{pmatrix}\cos\Phi&\sin\Phi&0\\-\sin\Phi&\cos\Phi&0\\0&0&1\end{pmatrix}$

7/9/18
R. P. Woodard

A2

Solutions for Asmt #4

(3a) * Start with the case $\ell=1$, $m=2$ & $n=3$ & make a column expansion

$$\begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix} = M_{11} \begin{vmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{vmatrix} - M_{21} \begin{vmatrix} M_{12} & M_{13} \\ M_{32} & M_{33} \end{vmatrix} + M_{31} \begin{vmatrix} M_{12} & M_{13} \\ M_{22} & M_{23} \end{vmatrix}$$

$$= M_{11}\left[M_{22}M_{33} - M_{32}M_{23}\right] + M_{21}\left[-M_{12}M_{33} + M_{32}M_{13}\right] + M_{31}\left[M_{12}M_{23} - M_{22}M_{13}\right]$$

$$= M_{11}\epsilon_{1jk}M_{j2}M_{k3} + M_{21}\epsilon_{2jk}M_{j2}M_{k3} + M_{31}\epsilon_{3jk}M_{j2}M_{k3} = \epsilon_{ijk}M_{i1}M_{j2}M_{k3}$$

* Now recall that interchanging any 2 ~~rows~~ columns gives a minus sign & $\det(M)=0$ if 2 columns same

$\Rightarrow$ $\underline{\epsilon_{ijk} M_{i\ell} M_{jm} M_{kn} = \det(M) * \epsilon_{\ell mn}}$

(3b) * Start with the case $i=1$, $j=2$ & $k=3$ & make a row expansion

$$\begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix} = M_{11} \begin{vmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{vmatrix} - M_{12} \begin{vmatrix} M_{21} & M_{23} \\ M_{31} & M_{33} \end{vmatrix} + M_{13} \begin{vmatrix} M_{21} & M_{22} \\ M_{31} & M_{32} \end{vmatrix}$$

$$= M_{11}\left[M_{22}M_{33} - M_{23}M_{32}\right] + M_{12}\left[-M_{21}M_{33} + M_{23}M_{31}\right] + M_{13}\left[M_{21}M_{32} - M_{22}M_{31}\right]$$

$$= M_{11}\epsilon_{1mn}M_{2m}M_{3n} + M_{12}\epsilon_{2mn}M_{2m}M_{3n} + M_{13}\epsilon_{3mn}M_{2m}M_{3n} = \epsilon_{\ell mn}M_{1\ell}M_{2m}M_{3n}$$

* Now recall that interchanging any 2 rows gives a minus, & that 2 identical rows $\Rightarrow 0$

$\Rightarrow$ $\epsilon_{\ell mn} M_{i\ell} M_{jm} M_{kn} = \det(M) * \epsilon_{ijk}$ ✓

(3c) * multiply (a) by $\epsilon_{\ell mn}$

$\Rightarrow$ $\epsilon_{ijk}\epsilon_{\ell mn} M_{i\ell} M_{jm} M_{kn} = \det(M)\,\epsilon_{\ell mn}\epsilon_{\ell mn} = \det(M)\left[\overbrace{\delta_{\ell\ell}\delta_{mm}}^{3\cdot 3} - \overbrace{\delta_{\ell m}\delta_{m\ell}}^{3}\right] = 6\det(M)$

(3d) * Left-inverse: $\underline{(M^{-1})_{\ell i} M_{ip}} = \dfrac{\epsilon_{ijk} M_{ip} M_{jm} M_{kn}}{2\det(M)}\,\epsilon_{\ell mn} = \frac{1}{2}\epsilon_{pmn}\epsilon_{\ell mn}$

$$= \tfrac{1}{2}\left(\delta_{p\ell}\delta_{mm} - \delta_{pm}\delta_{m\ell}\right) = \delta_{p\ell} \;✓$$

* Right-inverse: $M_{p\ell}(M^{-1})_{\ell i} = \dfrac{\epsilon_{\ell mn} M_{p\ell} M_{jm} M_{kn}}{2\det(M)}\,\epsilon_{ijk} = \frac{1}{2}\epsilon_{pjk}\epsilon_{ijk}$

$$= \tfrac{1}{2}\left(\delta_{pi}\delta_{jj} - \delta_{pj}\delta_{ji}\right) = \delta_{pi} \;✓$$

* NB all of these identities generalize to $N \times N$ matrices using the N-dimensional Levi-Civita density $\epsilon_{i_1 i_2 \cdots i_N}$