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R.P. Woodard

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Solutions to Asmt #3

(1) $* \; x = r\cos\phi = \frac{c\cos\phi}{1+\epsilon\cos\phi} \rightarrow \left(\frac{d+x}{a}\right)^2 = \frac{\left(\frac{d}{a}\right)^2 + 2\frac{d}{a}\left(\frac{c}{a}+\frac{d}{a}\epsilon\right)\cos\phi + \left(\frac{c}{a}+\frac{d}{a}\epsilon\right)^2\cos^2\phi}{[1+\epsilon\cos\phi]^2}$

$* \; y = r\sin\phi = \frac{c\sin\phi}{1+\epsilon\cos\phi} \rightarrow \left(\frac{y}{b}\right)^2 = \frac{(c/b)^2 - \left(\frac{c}{b}\right)^2\cos^2\phi}{[1+\epsilon\cos\phi]^2}$

(1a) If $\left(\frac{x+d}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ then we must have 3 conditions

$$\left.\begin{array}{l} (\alpha)\; \left(\frac{d}{a}\right)^2 + \left(\frac{c}{b}\right)^2 = 1 \\ (\beta)\; 2\frac{d}{a}\left(\frac{c}{a}+\frac{d\epsilon}{a}\right) = 2\epsilon \\ (\gamma)\; \left(\frac{c}{a}+\frac{d\epsilon}{a}\right)^2 = \epsilon^2 - \left(\frac{c}{b}\right)^2 \end{array}\right\} \rightarrow \left\{\begin{array}{l} (\alpha)+(\beta)+(\gamma) \rightarrow \left(\frac{d}{a}+\frac{c}{a}+\frac{d\epsilon}{a}\right)^2 = (1+\epsilon)^2 \rightarrow \frac{d}{a}+\frac{c}{a}+\frac{d\epsilon}{a} = 1+\epsilon \\ (\alpha)-(\beta)+(\gamma) \rightarrow \left(-\frac{d}{a}+\frac{c}{a}+\frac{d\epsilon}{a}\right)^2 = (1-\epsilon)^2 \rightarrow -\frac{d}{a}+\frac{c}{a}+\frac{d\epsilon}{a} = 1-\epsilon \\ \therefore \boxed{a = \frac{c}{1-\epsilon^2}} \;\&\; \boxed{d = \frac{\epsilon c}{1-\epsilon^2}} + (\alpha) \rightarrow \boxed{b = \frac{c}{\sqrt{1-\epsilon^2}}} \end{array}\right\}$$

(1b) $* \; \epsilon = 1 \rightarrow \left\{\begin{array}{l} x = \frac{c\cos\phi}{1+\cos\phi} = c\left[1 - \frac{1}{2}\sec^2(\phi/2)\right] \\ y = \frac{c\sin\phi}{1+\cos\phi} = c\tan(\phi/2) \end{array}\right\} \rightarrow \frac{x}{c} = 1 - \frac{1}{2}\left(1+\frac{y^2}{c^2}\right) \rightarrow \boxed{2\frac{x}{c} = 1 - \left(\frac{y}{c}\right)^2}$

* the x intercept is $y=0 \rightarrow \boxed{x = \frac{1}{2}c}$

* the y intercept is $x=0 \rightarrow \boxed{y = \pm c}$

(1c) If $\left(\frac{x+d}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$ then we must have 3 conditions

$$\left.\begin{array}{l} (\alpha)\; \left(\frac{d}{a}\right)^2 - \left(\frac{c}{b}\right)^2 = 1 \\ (\beta)\; 2\frac{d}{a}\left(\frac{c}{a}+\frac{d\epsilon}{a}\right) = 2\epsilon \\ (\gamma)\; \left(\frac{c}{a}+\frac{d\epsilon}{a}\right)^2 + \left(\frac{c}{b}\right)^2 = \epsilon^2 \end{array}\right\} \rightarrow \left\{\begin{array}{l} (\alpha)+(\beta)+(\gamma) \rightarrow \left(\frac{d}{a}+\frac{c}{a}+\frac{d\epsilon}{a}\right)^2 = (1+\epsilon)^2 \rightarrow \frac{d}{a}+\frac{c}{a}+\frac{d\epsilon}{a} = 1+\epsilon \\ (\alpha)-(\beta)+(\gamma) \rightarrow \left(\frac{d}{a}-\frac{c}{a}-\frac{d\epsilon}{a}\right)^2 = (1-\epsilon)^2 \rightarrow -\frac{d}{a}+\frac{c}{a}+\frac{d\epsilon}{a} = 1-\epsilon \\ \therefore \boxed{a = \frac{c}{1-\epsilon^2}} \;\&\; \boxed{d = \frac{\epsilon c}{1-\epsilon^2}} + (\alpha) \rightarrow \boxed{b = \frac{c}{\sqrt{\epsilon^2-1}}} \end{array}\right.$$

(2) $* \; \text{for } V(r) = -\frac{k}{r} \rightarrow \dot{r} = \pm\sqrt{\frac{2E}{m} + \frac{2k}{mr} - \frac{L^2}{m^2r^2}}$

(2a) $\left.\begin{array}{l} * \text{ parabola has } E = 0 \rightarrow \dot{r} = 0 \text{ at } r_p = L^2/2mk \\ * \text{ circle has } E = -\frac{mk^2}{2L^2} \rightarrow \dot{r} = 0 \text{ at } r_c = L^2/mk \end{array}\right\} \rightarrow \boxed{\frac{r_p}{r_c} = \frac{1}{2}}$

(2b) * to be at the same radius the parabolic orbit must be at $r_c = L^2/mk$

* for the parabola $v_p = \sqrt{\dot{r}^2 + r^2\dot{\phi}^2} = \sqrt{\frac{2k}{mr_c} - \cancel{\frac{L^2}{m^2r_c^2}} + \cancel{\frac{L^2}{m^2r_c^2}}} = \sqrt{2}\, k/L$

* for the circle $v_c = r_c\dot{\phi} = \frac{L}{mr_c} = k/L \rightarrow \boxed{v_p/v_c = \sqrt{2}}$

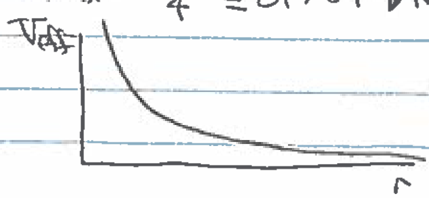
(3a) $\left.\begin{array}{l} * \; \dot{\phi} = \frac{L}{mr^2} \\ * \; \dot{r}^2 = \frac{2E}{m} - \frac{2V(r)}{m} - \frac{L^2}{m^2r^2} \end{array}\right\} \rightarrow \frac{r'^2}{r^4} \equiv u'^2 = \frac{2mE}{L^2} - \frac{2mV(1/u)}{L^2} - u^2$

(3b) $* \; V_{eff}(r) = \frac{L^2}{2mr^2} - \frac{k}{r}\exp[-r/a] \rightarrow \frac{\partial V_{eff}}{\partial r} = -\frac{L^2}{mr^3} + k\left(\frac{1}{r^2}+\frac{1}{ra}\right)e^{-r/a} = \frac{ka}{r^3}\left[-\frac{L^2}{mka} + \frac{r}{a}\left(1+\frac{r}{a}\right)e^{-\frac{r}{a}}\right]$

* NB max of $x(1+x)e^{-x}$ occurs at $x_{max} = \frac{\sqrt{17}-1}{4} \simeq 0.781 \rightarrow \max \simeq 0.637$

* for $\frac{L^2}{mka} > \max \rightarrow V_{eff}$ is monotonic

$\rightarrow$ only orbits are in from ∞ & back out



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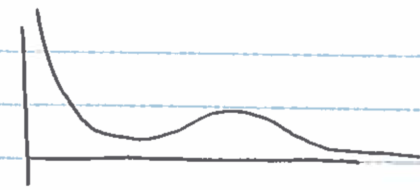
Solutions to Asmt #3



* for $\frac{L^2}{mka} < \max$ → $V_{\rm eff}$ has a local minima

→ orbits can be in from ∞ & back out
or bound in local minima

(3c) * $V(r) = -\frac{k}{r} e^{-r/a}$

* $\frac{\partial V}{\partial r} = \left[\frac{k}{r^2} + \frac{k}{ar}\right] e^{-r/a}$ → $\frac{2}{u_0^3} \frac{\partial V}{\partial r}\left(\frac{1}{u_0}\right) = \frac{2}{u_0^3}\left[k u_0^2 + \frac{k}{a} u_0\right] e^{-\frac{1}{a u_0}}$

* $\frac{\partial^2 V}{\partial r^2} = \left[-\frac{2k}{r^3} - \frac{2k}{ar^2} - \frac{k}{a^2 r}\right] e^{-r/a}$ → $\frac{1}{u_0^4} \frac{\partial^2 V}{\partial r^2}\left(\frac{1}{u_0}\right) = -\frac{1}{u_0^4}\left[2k u_0^3 + \frac{2k}{a} u_0^2 + \frac{k}{a^2} u_0\right] e^{-\frac{1}{a u_0}}$

$$\frac{2}{u_0^3} \frac{\partial V}{\partial r}\left(\frac{1}{u_0}\right) + \frac{1}{u_0^4} \frac{\partial^2 V}{\partial r^2}\left(\frac{1}{u_0}\right) = -\frac{k}{a u_0^3} e^{-\frac{1}{a u_0}}$$

∴ $K^2 = 1 + \frac{m}{L^2} * \frac{-k}{a u_0^3} e^{-\frac{1}{a u_0}}$

$\boxed{\text{Condition for Circular Orbit}}$ $\frac{m}{L^2} \frac{\partial V}{\partial r}\left(\frac{1}{u_0}\right) * \frac{1}{u_0^2} - u_0 = 0$

* $\frac{mk}{L^2} e^{-\frac{1}{a u_0}} = \frac{u_0^2}{\frac{1}{a} + u_0}$ → $K^2 = 1 - \frac{1}{a u_0} \frac{1}{1 + a u_0} = 1 - \frac{(R/a)^2}{1 + R/a}$

* $\frac{1}{K} \simeq 1 + \frac{1}{2} \frac{(R/a)^2}{1 + R/a}$ → apsides advance by $\frac{2\pi}{K} - 2\pi \simeq \frac{\pi (R/a)^2}{1 + R/a}$