8/22/18
R.P. Woodard

Solutions to Asmt #2

(1a) * $S[q] = \int_{t_1}^{t_2} dt' \, L\big(q(t'), \dot{q}(t'), \ddot{q}(t'), t'\big)$

* $\frac{\delta S}{\delta q(t)} = \int_{t_1}^{t_2} dt' \Big\{ \frac{\partial L}{\partial q} \delta(t'-t) + \frac{\partial L}{\partial \dot{q}} \delta'(t'-t) + \frac{\partial L}{\partial \ddot{q}} \delta''(t'-t) \Big\} = \frac{\partial L}{\partial q} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) + \frac{d^2}{dt^2}\Big(\frac{\partial L}{\partial \ddot{q}}\Big) = 0$

(1b) * $\frac{dE}{dt} = \ddot{q}\Big[\cancel{\frac{\partial L}{\partial \dot{q}}} - \cancel{\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}}\Big] + \dot{q}\Big[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}}\Big] + \cancel{\dddot{q}\frac{\partial L}{\partial \ddot{q}}} + \cancel{\ddot{q}\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}} - \frac{\partial L}{\partial q}\dot{q} - \cancel{\frac{\partial L}{\partial \dot{q}}\ddot{q}} - \cancel{\frac{\partial L}{\partial \ddot{q}}\dddot{q}}$

$= \dot{q}\Big[-\frac{\partial L}{\partial q} + \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}}\Big] = 0$ QED

(1c) * $L = L(q, \dot{q}, \ddot{q}) \rightarrow \dot{q}\frac{\partial L}{\partial \dot{q}}, \ddot{q}\frac{\partial L}{\partial \ddot{q}}$ & $-L$ each depend only upon $q, \dot{q}$ & $\ddot{q}$, not $\dddot{q}$

* $-\dot{q}\frac{d}{dt}\Big(\frac{\partial L}{\partial \ddot{q}}\Big) = -\dot{q}\Big\{\frac{\partial^2 L}{\partial q \partial \ddot{q}}\dot{q} + \frac{\partial^2 L}{\partial \dot{q}\partial \ddot{q}}\ddot{q} + \frac{\partial^2 L}{\partial \ddot{q}^2}\dddot{q}\Big\}$

$\therefore$ E is linear in $\dddot{q}$

* Hence if $\frac{\partial^2 L}{\partial \ddot{q}^2} \neq 0$ E can never be bounded below

* NB this is why Newton was right about $F = ma$

(2a) * $\dot{q}(t) = -\sum_{n=0}^{\infty} \frac{n\pi}{T} a_n \sin\Big(\frac{n\pi t}{T}\Big) \rightarrow \dot{q}^2(t) = \sum_{k=0}^{\infty} \frac{k\pi}{T} a_k \sum_{l=0}^{\infty} \frac{l\pi}{T} a_l \sin\Big(\frac{k\pi t}{T}\Big)\sin\Big(\frac{l\pi t}{T}\Big)$

* $\sin(\alpha)\sin(\beta) = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$

$\rightarrow \sin\Big(\frac{k\pi t}{T}\Big)\sin\Big(\frac{l\pi t}{T}\Big) = \frac{d}{dt}\Big\{\frac{T}{2(k-l)\pi}\sin\Big[\frac{(k-l)\pi t}{T}\Big] - \frac{T}{2(k+l)\pi}\sin\Big[\frac{(k+l)\pi t}{T}\Big]\Big\}$

* the integral of this $\int_0^T dt = \frac{T}{2}\delta_{kl} \rightarrow \int_0^T dt \frac{1}{2} m \dot{q}^2 = \frac{T}{4} m \sum_{k=0}^{\infty}\Big(\frac{k\pi}{T}\Big)^2 a_k^2$

* $\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)$

$\rightarrow \int_0^T dt \cos\Big(\frac{k\pi t}{T}\Big)\cos\Big(\frac{l\pi t}{T}\Big) = \frac{T}{2}\delta_{kl} \rightarrow -\frac{1}{2}k\int_0^T dt\, q^2 = -\frac{T}{4}k\sum_{k=0}^{\infty} a_k^2$

$\therefore S[q] = \frac{T}{4}\sum_{n=0}^{\infty} a_n^2\Big[m\Big(\frac{n\pi}{T}\Big)^2 - k\Big]$

(2b) $\frac{\partial S}{\partial a_n} = \frac{T}{2}\Big[m\Big(\frac{n\pi}{T}\Big)^2 - k\Big] a_n = 0$

(2c) * $\frac{\delta S}{\delta q(t)} = -m\ddot{q}(t) - k\,q(t) = 0$

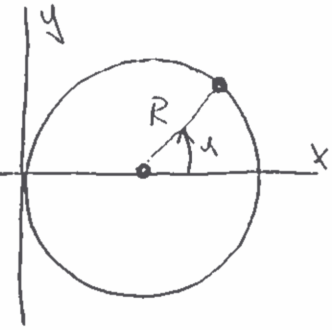
$\rightarrow \sum_{n=0}^{\infty}\Big[m\Big(\frac{n\pi}{T}\Big)^2 - k\Big] a_n \cos\Big(\frac{n\pi t}{T}\Big) = 0$

$\rightarrow \Big[m\Big(\frac{n\pi}{T}\Big)^2 - k\Big] a_n = 0$ (same eqn!)

R.P. Woodard

A3

Solutions to Asmt #2



(3a) * $x(t) = R + R\cos(\alpha(t)) \rightarrow \dot{x} = -R\dot{\alpha}\sin(\alpha)$

* $y(t) = 0 + R\sin(\alpha(t)) \rightarrow \dot{y} = +R\dot{\alpha}\cos(\alpha)$

* $L_z = m(x\dot{y} - y\dot{x}) = mR^2\dot{\alpha}(1+\cos(\alpha)) = 2mR^2\dot{\alpha}\cos^2(\alpha/2)$

(3b) * $E = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + V\left(\sqrt{x^2+y^2}\right)$

* $\frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}mR^2\dot{\alpha}^2$

* $x^2+y^2 = R^2[2+2\cos(\alpha)] = 4R^2\cos^2(\alpha/2) \rightarrow \sqrt{x^2+y^2} = 2R\cos(\alpha/2)$

$\therefore$ $E = \frac{1}{2}mR^2\dot{\alpha}^2 + V\left(2R\cos(\alpha/2)\right)$

(3c) * $L = 2mR^2\dot{\alpha}\cos^2(\alpha/2) \rightarrow \dot{\alpha} = \frac{L}{2mR^2\cos^2(\alpha/2)} = \frac{2L}{m} * \left[\frac{1}{2R\cos(\alpha/2)}\right]^2$

* $E = \frac{2R^2L^2}{m}\left[\frac{1}{2R\cos(\alpha/2)}\right]^4 + V\left(2R\cos(\alpha/2)\right)$

* the only way for E to be constant is for $E = 0 \rightarrow V(r) = -\frac{2R^2L^2}{m}\frac{1}{r^4}$

(3d) * $dt = \frac{d\alpha}{\dot{\alpha}} = \frac{mR^2}{L}(1+\cos(\alpha))\,d\alpha = \frac{mR^2}{L}d\left[\alpha + \sin(\alpha)\right]$

* $T = \frac{mR^2}{L} * 2\pi$