11/15/18
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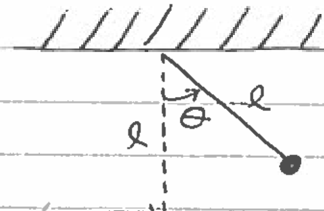
Solutions to Asmt #13



(1a) $* L = \frac{1}{2} m \ell^2 \dot{\theta}^2 - mg\ell(1-\cos\theta)$

$* p_\theta = m\ell^2\dot{\theta} \rightarrow H = \frac{p_\theta^2}{2m\ell^2} + mg\ell(1-\cos\theta)$

$* E = mg\ell(1-\cos\theta_0) \rightarrow$ full range is $-\theta_0 \leq \theta \leq \theta_0 = 2\sin^{-1}\left(\sqrt{\frac{E}{2mg\ell}}\right)$

$* J(E) = 2\int_{-\theta_0}^{\theta_0} d\theta \sqrt{2m^2 g \ell^3}\sqrt{\cos\theta - \cos\theta_0}$ where $\cos\theta_0 = 1 - E/mg\ell$

$* dJ = m\ell\sqrt{2g\ell}\int_{-\theta_0}^{\theta_0} d\theta \frac{dE/mg\ell}{\sqrt{\cos\theta-\cos\theta_0}} \rightarrow f = \frac{dE}{dJ} = \sqrt{\frac{g}{2\ell}}\left[\int_{-\theta_0}^{\theta_0} d\theta \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\right]^{-1}$

(1b) $*$ for small $\theta_0 \geq \theta \rightarrow \cos\theta - \cos\theta_0 \simeq \frac{1}{2}(\theta_0^2 - \theta^2)$

$* \int_{-\theta_0}^{\theta_0} d\theta \frac{1}{\sqrt{\cos\theta-\cos\theta_0}} \simeq \sqrt{2}\int_{-\theta_0}^{\theta_0} d\theta \frac{1}{\sqrt{\theta_0^2-\theta^2}} = \sqrt{2}\,\pi \rightarrow f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$

(1c) $*$ to next order $\rightarrow \cos\theta - \cos\theta_0 \simeq \frac{1}{2}(\theta_0^2-\theta^2) - \frac{1}{24}(\theta_0^4-\theta^4) + \dots = \frac{1}{2}(\theta_0^2-\theta^2)\left[1 - \frac{1}{12}(\theta_0^2+\theta^2) + \dots\right]$

$* \int_{-\theta_0}^{\theta_0} d\theta \frac{1}{\sqrt{\cos\theta-\cos\theta_0}} \simeq \sqrt{2}\int_{-\theta_0}^{\theta_0} d\theta \frac{1}{\sqrt{\theta_0^2-\theta^2}}\left\{1 + \frac{1}{24}(\theta_0^2+\theta^2) - \dots\right\} = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha \left\{1 + \frac{\theta_0^2}{24}\left[1+\sin^2(\alpha)\right] + \dots\right\}$

$\rightarrow f(\theta_0) = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}\left[1 - \frac{\theta_0^2}{16} + O(\theta_0^4)\right]$ $\qquad = \sqrt{2}\,\pi\left\{1 + \frac{\theta_0^2}{16} + O(\theta_0^4)\right\}$

(2a) $* \frac{\delta S}{\delta q} = -\frac{m}{\omega^2}\left[g\overset{\cdot\cdot\cdot\cdot}{q} + \omega^2\ddot{q} + \omega^4 q\right] = 0 \qquad q(t) \propto e^{\pm ikt} \rightarrow gk^4 - \omega^2 k^2 + \omega^4 = 0 \rightarrow k_\pm^2 = \frac{\omega^2}{2g}\left[1 \pm \sqrt{1-4g}\right]$

$* q(t) = \frac{g/\omega^2}{\sqrt{1-4g}}\left\{-(k_-^2 q_0 + \ddot{q}_0)\cos(k_+ t) - \frac{(k_-^2\dot{q}_0 + \dddot{q}_0)}{k_+}\sin(k_+ t) + (k_+^2 q_0 + \ddot{q}_0)\cos(k_- t) + \frac{(k_+^2\dot{q}_0 + \dddot{q}_0)}{k_-}\sin(k_- t)\right\}$

(2b) $* q(t) = q^{(0)}(t) + g\, q^{(1)}(t) + \dots \rightarrow \left\{\begin{array}{l} \ddot{q}^{(0)} + \omega^2 q^{(0)} = 0 \\ \ddot{q}^{(1)} + \omega^2 q^{(1)} = -\frac{g}{\omega^2}\overset{\cdot\cdot\cdot\cdot}{q}^{(0)} \\ \ddot{q}^{(2)} + \omega^2 q^{(2)} = -\frac{g}{\omega^2}\overset{\cdot\cdot\cdot\cdot}{q}^{(1)} \\ \text{etcetera} \end{array}\right\}$

$* q^{(0)}(t) = q_0\cos(\omega t) + \frac{\dot{q}_0}{\omega}\sin(\omega t)$

$* \ddot{q}^{(1)} + \omega^2 q^{(1)} = +g\ddot{q}^{(0)} = -g\omega^2 q^{(0)} \rightarrow q^{(1)}(t) = \int_0^t dt' \frac{\sin[\omega(t-t')]}{\omega} \times -g\omega^2 q^{(0)}(t')$

$*$ performing the integrations gives

$q^{(1)}(t) = \frac{1}{2} g t \times \dot{q}^{(0)}(t) - \frac{1}{2} g \frac{\dot{q}_0}{\omega}\sin(\omega t)$ $\rightarrow$ both consistent with $\omega \rightarrow \omega\left[1 + \frac{1}{2}g + \dots\right]$

(2c) $*$ perturbative solution recovers only $k_-^2 = \frac{\omega^2}{2g}\left[1 - \sqrt{1-4g}\right] = \omega^2\left[1 + g + 2g^2 + \dots\right]$

$\rightarrow q_{pert}(t) = q_0\cos(k_- t) + \frac{\dot{q}_0}{k_-}\sin(k_- t) \qquad$ NB $\ddot{q}_{pert} = -k_-^2 q_{pert}$

(3a) $* H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 + \frac{1}{4}\lambda q^4 = E \rightarrow q^2(E) = \frac{m\omega^2}{\lambda}\left[\sqrt{1 + \frac{4\lambda E}{m^2\omega^4}} - 1\right]$

$* J = 2\int_{-q(E)}^{q(E)} dq \sqrt{2mE - m^2\omega^2 q^2 - \frac{1}{2}\lambda m q^4} \rightarrow f = \frac{dE}{dJ} = \frac{1}{2m}\left[\int_{-q(E)}^{+q(E)} \frac{dq}{\sqrt{2mE - m^2\omega^2 q^2 - \frac{1}{2}\lambda m q^4}}\right]^{-1}$

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(3b) * $\epsilon \equiv \frac{\lambda E}{m^2\omega^4}$

* $q \equiv \sqrt{\frac{2E}{m\omega^2}}\sin(\alpha) \rightarrow$ upper limit is $\alpha(\epsilon) = \sin^{-1}\Bigl[\sqrt{\frac{1}{2\epsilon}(\sqrt{1+4\epsilon}-1)}\Bigr] = \frac{\pi}{2} - \sqrt{\epsilon}\bigl(1 - \frac{5}{6}\epsilon + \dots$

$$\rightarrow f(\epsilon) = \frac{1}{2}\omega\Bigl[\int_{-\alpha(\epsilon)}^{+\alpha(\epsilon)} \frac{d(\sin(\alpha))}{\sqrt{1-\sin^2(\alpha)-\epsilon\sin^4(\alpha)}}\Bigr]^{-1}$$

* make one final change of variables $\sin(\alpha) \equiv \sin[\alpha(\epsilon)] * \sin(\beta)$

$$\rightarrow f(\epsilon) = \frac{1}{2}\omega\Bigl[\frac{1}{(1+4\epsilon)^{\frac{1}{4}}}\int_{-\frac{\pi}{2}}^{+\pi/2} d\beta \frac{1}{\sqrt{1-\frac{1}{2}(1-\frac{1}{\sqrt{1+4\epsilon}})\cos^2(\beta)}}\Bigr]^{-1} = \boxed{\frac{\omega}{2\pi}\Bigl[1 + \frac{3}{4}\epsilon + \mathcal{O}(\epsilon^2)\Bigr]}$$

(3c) * exact eqn is $\ddot{q} + \omega^2 q = -\frac{\lambda}{m}q^3$

* $q(t) = q^{(0)}(t) + \frac{\lambda}{m}q^{(1)}(t) + \dots \rightarrow \left\{\begin{array}{l}\ddot{q}^{(0)} + \omega^2 q^{(0)} = 0 \\ \ddot{q}^{(1)} + \omega^2 q^{(1)} = -[q^{(0)}]^3\end{array}\right\}$

* $q^{(0)}(t) = q_0\cos(\omega t) + \frac{\dot{q}_0}{\omega}\sin(\omega t) = Ae^{-i\omega t} + A^* e^{i\omega t}$ where $A = \frac{1}{2}[q_0 + \frac{i\dot{q}_0}{\omega}]$

* $q^{(1)}(t) = -\int_0^t dt' \frac{\sin[\omega(t-t')]}{\omega}[q^{(0)}(t')]^3 = \frac{-1}{2i\omega}\int_0^t dt' \Bigl[e^{i\omega(t-t')} - e^{-i\omega(t-t')}\Bigr]\Bigl[Ae^{-i\omega t'} + A^* e^{i\omega t'}\Bigr]^3$

* performing the integrations gives

$$q^{(1)}(t) = \underbrace{\frac{3AA^*}{2i\omega}\Bigl[Ae^{-i\omega t} - A^*e^{i\omega t}\Bigr]t}_{= +\frac{3AA^*}{2\omega^2}\dot{q}^{(0)}(t) * t} + \frac{e^{-i\omega t}}{8\omega^2}\Bigl[-2A^3 - 6A^2A^* + 6AA^{*2} + A^{*3}\Bigr] + \frac{A^3}{8\omega^2}e^{-3i\omega t}$$
$$+ \frac{e^{i\omega t}}{8\omega^2}\Bigl[A^3 + 6A^2A^* - 6AA^{*2} - 2A^{*3}\Bigr] + \frac{A^{*3}}{8\omega^2}e^{3i\omega t}$$

frequency shift   amplitude shift   higher harmonics