11/12/18
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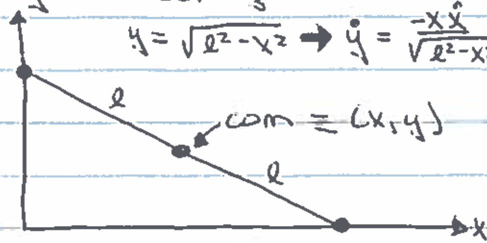
Solutions to Asmt #12

$I_{com} = \frac{1}{3} m \ell^3$

$y = \sqrt{\ell^2 - x^2} \rightarrow \dot{y} = \frac{-x\dot{x}}{\sqrt{\ell^2 - x^2}}$



$\tan(\theta) = \frac{y}{x} = \sqrt{\frac{\ell^2}{x^2} - 1} \rightarrow \dot{\theta} = \frac{-\dot{x}}{\sqrt{\ell^2 - x^2}}$

① $*\ KE = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I_{com} \dot{\theta}^2 = \frac{2}{3} m \ell^2 \dot{\theta}^2$

$*\ PE = mgy = mg\ell \sin(\theta)$

$\therefore L = \frac{2}{3} m \ell^2 \dot{\theta}^2 - mg\ell \sin(\theta)$

$*\ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{4}{3} m \ell^2 \dot{\theta} \rightarrow H(\theta, p_\theta) = \frac{3 p_\theta^2}{8 m \ell^2} + mg\ell \sin\theta$

$*\ H(\theta, \frac{\partial W}{\partial \theta}) = E \rightarrow \frac{\partial W}{\partial \theta} = \pm \sqrt{\frac{8}{3} m \ell^2 E - \frac{8}{3} m^2 g \ell^3 \sin\theta} \rightarrow W(\theta, E) = \sqrt{\frac{8}{3} m \ell^2 E} \int_0^\theta d\phi \sqrt{1 - \frac{mg\ell}{E} \sin\phi}$

$*$ NB this can be express as an elliptic integral

②a $*\ KE = \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) = \frac{1}{2} m a^2 [\sinh^2(v) + \sin^2(u)](\dot{v}^2 + \dot{u}^2) + \frac{1}{2} m a^2 \sinh^2(v) \sin^2(u) \dot{\phi}^2$

$\rightarrow p_v = m a^2 [\sinh^2(v) + \sin^2(u)] \dot{v}$, $p_u = m a^2 [\sinh^2(v) + \sin^2(u)] \dot{u}$ & $p_\phi = m a^2 \sinh^2(v) \sin^2(u) \dot{\phi}$

$\therefore H = \frac{p_v^2 + p_u^2}{2 m a^2 [\sinh^2(v) + \sin^2(u)]} + \frac{p_\phi^2}{2 m a^2 \sinh^2(v) \sin^2(u)} + V(v, u, \phi)$

$*$ Separable for $V = \frac{V_v(v)}{\sinh^2(v) + \sin^2(u)} + \frac{V_u(u)}{\sinh^2(v) + \sin^2(u)} + \frac{V_\phi(\phi)}{\sinh^2(v) \sin^2(u)}$

②b $*\ V = -\frac{G m_1}{a[\cosh(v) - \cos(u)]} - \frac{G m_2}{a[\cosh(v) + \cos(u)]} = \frac{-G(m_1+m_2)\cosh(v) - G(m_1 - m_2)\cos(u)}{a[\cosh^2(v) - \cos^2(u)]} = \frac{\text{same}}{a[\sinh^2(v) + \sin^2(u)]}$

$*\ W = W_v(v, \vec{\alpha}) + W_u(u, \vec{\alpha}) + W_\phi(\phi, \vec{\alpha})$

$\rightarrow \frac{(\frac{\partial W_v}{\partial v})^2 + (\frac{\partial W_u}{\partial u})^2}{2 m a^2 [\sinh^2(v) + \sin^2(u)]} + \frac{(\frac{\partial W_\phi}{\partial \phi})^2}{2 m a^2 \sinh^2(v) \sin^2(u)} - \frac{G(m_1+m_2)\cosh(v) + G(m_1 - m_2)\cos(u)}{a[\sinh^2(v) + \sin^2(u)]} = E$

$\rightarrow (\frac{\partial W_v}{\partial v})^2 + (\frac{\partial W_u}{\partial u})^2 + (\frac{\partial W_\phi}{\partial \phi})^2 [\frac{1}{\sinh^2(v)} + \frac{1}{\sin^2(u)}] - 2G(m_1+m_2) m a \cosh(v) - 2G(m_1 - m_2) m a \cos(u) = 2 m a^2 E [\sinh^2(v) + \sin^2(u)]$

$W_v(v, E, \alpha_u, \alpha_\phi) = \int dv \sqrt{2 m a^2 E \sinh^2(v) + 2 m a G(m_1+m_2)\cosh(v) - \frac{\alpha_\phi^2}{\sinh^2(v)} - \alpha_v^2}$

$W_u(u, E, \alpha_u, \alpha_\phi) = \int du \sqrt{2 m a^2 E \sin^2(u) + 2 m a G(m_1 - m_2)\cos(u) - \frac{\alpha_\phi^2}{\sin^2(u)} + \alpha_v^2}$

$W_\phi(\phi, \alpha_\phi) = \alpha_\phi \phi$

③ $*\ \dot{x} = 2\ell [1 + \cos(2\lambda)] \dot{\lambda} = 4\ell \cos^2(\lambda) \dot{\lambda}$

$*\ \dot{y} = 2\ell \sin(2\lambda) \dot{\lambda} = 4\ell \cos(\lambda) \sin(\lambda) \dot{\lambda}$

$\rightarrow KE = 8 m \ell^2 \cos^2(\lambda) \dot{\lambda}^2$

$*\ PE = 2mg\ell \sin^2(\lambda)$

$\rightarrow L = 8 m \ell^2 \cos^2(\lambda) \dot{\lambda}^2 - 2mg\ell \sin^2(\lambda)$

$*\ p_\lambda = 16 m \ell^2 \cos^2(\lambda) \dot{\lambda} \rightarrow H(\lambda, p_\lambda) = \frac{p_\lambda^2}{32 m \ell^2 \cos^2(\lambda)} + 2mg\ell \sin^2(\lambda) \rightarrow$ full range is $-\sin^{-1}(\sqrt{\frac{E}{2mg\ell}}) \le \lambda \le +\sin^{-1}(\ldots)$, $\equiv \lambda(E)$

$*\ J(E) = 2 \int_{-\lambda(E)}^{+\lambda(E)} d\lambda \sqrt{32 E m \ell^2} \cos(\lambda) \sqrt{1 - \frac{2mg\ell}{E} \sin^2(\lambda)} = 8E \sqrt{\frac{\ell}{g}} \int_{-\pi/2}^{\pi/2} d\alpha \cos^2(\alpha) = 4\pi E \sqrt{\frac{\ell}{g}} \rightarrow E = \sqrt{\frac{g}{\ell}} \frac{J}{4\pi}$

$\rightarrow f = \frac{dE}{dJ} = \frac{1}{4\pi} \sqrt{\frac{g}{\ell}}$