10/19/18
R.P. Woodard

Solutions to Assignment #10

(1a) * suppose $P_1 = P_1(q_1, q_2, p_1, p_2)$ then $1 = \{Q_1, P_1\} = 2q_1 \frac{\partial P_1}{\partial p_1}$ & $0 = \{Q_2, P_1\} = \frac{\partial P_1}{\partial p_1} + \ldots$

$$\Rightarrow P_1 = \left(\frac{p_1 - p_2}{2q_1}\right) + f(q_1, q_2)$$

* suppose $P_2 = P_2(q_1, q_2, p_1, p_2)$ then $0 = \{Q_1, P_2\} = \underset{\substack{\vee\\ 2q_1}}{\ }\frac{\partial P_2}{\partial p_1}$ & $1 = \{Q_2, P_2\} = \frac{\partial P_2}{\partial p_1} + \frac{\partial P_2}{\partial p_2}$

$$\Rightarrow P_2 = p_2 + g(q_1, q_2)$$

* $0 = \{P_1, P_2\} = \frac{1}{2q_1}\left(\frac{\partial g}{\partial q_1} - \frac{\partial g}{\partial q_2}\right) + \frac{\partial f}{\partial q_2} \Rightarrow \left\{\begin{array}{l} g(q_1, q_2) = G(q_1 + q_2) + h(q_2) \\ f(q_1, q_2) = \frac{1}{2q_1} h(q_2) \end{array}\right\}$

$$\therefore P_1 = \left(\frac{p_1 - p_2}{2q_1}\right) + \frac{1}{2q_1} h(q_2) \quad \& \quad P_2 = p_2 + G(q_1 + q_2) + h(q_2)$$

(1b) * $\left\{\begin{array}{l} P_1 = \left(\frac{p_1 - p_2}{2q_1}\right) \\ P_2 = p_2 + \frac{c}{b}(q_1 + q_2)^2 \end{array}\right\} \Rightarrow H = \frac{a}{2} P_1^2 + b P_2$

* $Q_1(t) = Q_1(0) + a P_1(0) t \qquad P_1(t) = P_1(0)$

* $Q_2(t) = Q_2(0) + bt \qquad P_2(t) = P_2(0)$

(1c) * $q_1(t) = \sqrt{Q_1(t)} = \left[q_1^2(0) + a\left(\frac{p_1(0) - p_2(0)}{2q_1(0)}\right)t\right]^{\frac{1}{2}}$

* $q_2(t) = Q_2(t) - \sqrt{Q_1(t)} = q_1(0) + q_2(0) + bt - \left[q_1^2(0) + a\left(\frac{p_1(0) - p_2(0)}{2q_1(0)}\right)t\right]^{\frac{1}{2}}$

* $p_1(t) = \left[1 + a\left(\frac{p_1(0) - p_2(0)}{2q_1^3(0)}\right)t\right]^{\frac{1}{2}}\left[p_1(0) - p_2(0)\right] + p_2(0) + \frac{c}{b}\left[q_1(0) + q_2(0)\right]^2 - \frac{c}{b}\left[q_1(0) + q_2(0) + bt\right]^2$

* $p_2(t) = p_2(0) + \frac{c}{b}\left[q_1(0) + q_2(0)\right]^2 - \frac{c}{b}\left[q_1(0) + q_2(0) + bt\right]^2$

* A little tough to get directly!

(2a) * $\cancel{B}\ \vec{\nabla} \times \vec{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\right) \times \frac{B_0}{2}\left(-y\hat{x} + x\hat{y}\right) = B_0 \hat{z}$ ✓

(2b) * $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{qB_0}{2}(-\dot{x}y + \dot{y}x)$

* $p_x = m\dot{x} - \frac{1}{2}qB_0 y \Rightarrow \dot{x} = \frac{p_x}{m} + \frac{qB_0}{2m}y$

* $p_y = m\dot{y} + \frac{1}{2}qB_0 x \Rightarrow \dot{y} = \frac{p_y}{m} - \frac{qB_0}{2m}x$

$$\Rightarrow H = \frac{1}{2m}\left[\left(p_x + \tfrac{1}{2}qB_0 y\right)^2 + \left(p_y - \tfrac{1}{2}qB_0 x\right)^2\right]$$

(2c) * $\{x, y\} = \frac{1}{\alpha^2}\left\{\sqrt{2P_1}\sin(Q_1) + P_2, \sqrt{2P_1}\cos(Q_1) + Q_2\right\} = \frac{1}{\alpha^2}\left[\cos^2(Q_1) + \sin^2(Q_1) - 1\right] = 0$ ✓

* $\{x, p_x\} = \frac{1}{2}\left\{\sqrt{2P_1}\sin(Q_1) + P_2, \sqrt{2P_1}\cos(Q_1) - Q_2\right\} = \frac{1}{2}\left[\cos^2(Q_1) + \sin^2(Q_1) + 1\right] = 1$ ✓

* $\{x, p_y\} = \frac{1}{2}\left\{\sqrt{2P_1}\sin(Q_1) + P_2, -\sqrt{2P_1}\sin(Q_1) + P_2\right\} = \frac{1}{2}\left[-\cos(Q_1)\sin(Q_1) + \sin(Q_1)\cos(Q_1)\right] = 0$ ✓

* $\{y, p_x\} = \frac{1}{2}\left\{\sqrt{2P_1}\cos(Q_1) + Q_2, \sqrt{2P_1}\cos(Q_1) - Q_2\right\} = \frac{1}{2}\left[-\sin(Q_1)\cos(Q_1) + \cos(Q_1)\sin(Q_1)\right] = 0$ ✓

* $\{y, p_y\} = \frac{1}{2}\left\{\sqrt{2P_1}\cos(Q_1) + Q_2, -\sqrt{2P_1}\sin(Q_1) + P_2\right\} = \frac{1}{2}\left[+\sin^2(Q_1) + \cos^2(Q_1) + 1\right] = 1$ ✓

* $\{p_x, p_y\} = \frac{\alpha^2}{4}\left\{\sqrt{2P_1}\cos(Q_1) - Q_2, -\sqrt{2P_1}\sin(Q_1) + P_2\right\} = \frac{\alpha^2}{4}\left[+\sin^2(Q_1) + \cos^2(Q_1) - 1\right] = 0$ ✓

(2d) * choose $\alpha = \sqrt{qB_0}$

* $p_x + \frac{qB_0}{2}y = \sqrt{2qB_0 P_1}\cos(Q_1)$

* $p_y - \frac{qB_0}{2}x = -\sqrt{2qB_0 P_1}\sin(Q_1)$

$$\Rightarrow H = \frac{qB_0}{m}P_1 \Rightarrow \quad \dot{Q}_1 = \frac{qB_0}{m} \quad \dot{P}_1 = 0 \quad \dot{Q}_2 = 0 \quad \dot{P}_2 = 0$$

10/20/18
R.P. Woodard

Solutions to Assignment #10

(2e) * define $\omega_B \equiv \frac{qB_0}{m}$ (the cyclotron frequency)

* the general solution for the new coordinates is

$$Q_1(t) = Q_1(0) + \omega_B t \qquad P_1(t) = P_1(0)$$

$$Q_2(t) = Q_2(0) \qquad P_2(t) = P_2(0)$$

$$\rightarrow x(t) = \sqrt{\frac{2P_1(0)}{qB_0}} \sin\Big(Q_1(0) + \omega_B t\Big) + \frac{P_2(0)}{\sqrt{qB_0}} = x_0 + \frac{\dot{y}_0}{\omega_B}\Big[1 - \cos(\omega_B t)\Big] + \frac{\dot{x}_0}{\omega_B}\sin(\omega_B t)$$

$$\rightarrow y(t) = \sqrt{\frac{2P_1(0)}{qB_0}} \cos\Big(Q_1(0) + \omega_B t\Big) + \frac{Q_2(0)}{\sqrt{qB_0}} = y_0 - \frac{\dot{x}_0}{\omega_B}\Big[1 - \cos(\omega_B t)\Big] + \frac{\dot{y}_0}{\omega_B}\sin(\omega_B t)$$

(3a) $* \dot{q} = \frac{\partial H}{\partial p} = p q^4$

$* \dot{p} = -\frac{\partial H}{\partial q} = \frac{1}{q^3} - 2p^2 q^3$

$$\Big\} \rightarrow \frac{\ddot{q}}{q^4} - \frac{2\dot{q}^2}{q^5} - \frac{1}{q^3} = 0 \rightarrow \frac{d^2}{dt^2}\Big(\frac{1}{q}\Big) + \frac{1}{q} = 0$$

(3b) $* Q = \frac{1}{q}$

$* P = -q^2 p$

$$\Big\} \rightarrow \{Q, P\} = \Big\{\frac{1}{q}, -q^2 p\Big\} = 1 \checkmark \ \& \ H = \tfrac{1}{2}Q^2 + \tfrac{1}{2}P^2$$

(3c) * the general initial value solution is

$$Q(t) = Q_0 \cos(t) + P_0 \sin(t) = \frac{1}{q(t)} \rightarrow \frac{d^2}{dt^2}\Big(\frac{1}{q}\Big) + \frac{1}{q} = 0 \checkmark$$

$$P(t) = P_0 \cos(t) - Q_0 \sin(t)$$