PHY 6246 October 14, 2019

Assignment #9

Reading: Chapter 8 in *Goldstein*.

Problems: Due by the start of class on Monday, 10/28/19.

(1) Recall the higher derivative Lagrangian $L(q, \dot{q}, \ddot{q})$ of problem 1 on assignment #2 whose Euler-Lagrange equation is,

$$\frac{d^2}{dt^2}\Big(\frac{\partial L}{\partial \ddot{q}}\Big) - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) + \frac{\partial L}{\partial q} = 0 \; .$$

Because this system has four pieces of initial value data, it requires two canonical coordinates. They are usually taken to be,

$$Q_1 \equiv q \qquad , \qquad P_1 \equiv \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \ddot{q}}\Big) \; ,$$
$$Q_2 \equiv \dot{q} \qquad , \qquad P_2 \equiv \frac{\partial L}{\partial \ddot{q}} \; .$$

Assume that the system is *non-degenerate*, that is, one can invert the equation for P_2 to solve for $\ddot{q}$ as a function $a(Q_1, Q_2, P_2)$. Legendre transformation gives the Hamiltonian,

$$H \equiv \sum_{n=1}^{2} P_n \frac{d^n q}{dt^n} - L = P_1 Q_2 + P_2 a(Q_1, Q_2, P_2) - L\Big(Q_1, Q_2, a(Q_1, Q_2, P_2)\Big) \; .$$

(a) Prove the Hamilton equation $\dot{Q}_1 = \frac{\partial H}{\partial P_1}$ reproduces the definition of Q_2 above.

(b) Prove the Hamilton equation $\dot{Q}_2 = \frac{\partial H}{\partial P_2}$ reproduces the relation $\ddot{q} = a(Q_1, Q_2, P_2)$.

(c) Prove the Hamilton equation $\dot{P}_2 = -\frac{\partial H}{\partial Q_2}$ reproduces the definition of P_1 above.

(d) Prove the Hamilton equation $\dot{P}_1 = -\frac{\partial H}{\partial Q_1}$ reproduces the Euler-Lagrange equation.

(e) Prove the Hamiltonian is conserved if the Lagrangian has no explicit time dependence.

(2) Recall the damped harmonic oscillator of problem 3 on assignment #1 whose Lagrangian is,

$$L = \frac{1}{2} e^{\gamma t}\Big(m\dot{q}^2 - kq^2\Big) \; .$$

(a) Find the canonical momentum and construct the Hamiltonian.

(b) Solve the Hamiltonian evolution equations for $Q(t)$ and $P(t)$ starting from $Q(0)$ and $P(0)$.

(c) Use your solutions to express the energy as a function of time, $Q(0)$ and $P(0)$.

(3) Recall the Lagrangian for a point particle $\vec{r}(t)$ of mass m and charge q which moves in pacetime dependent lectric potential V and vector potential $\vec{A}$,

$$L = -mc\sqrt{c^2 - \|\dot{\vec{r}}\|^2} - qV\Big(t, \vec{r}(t)\Big) + q\dot{\vec{r}} \cdot \vec{A}\Big(t, \vec{r}(t)\Big) \ .$$

(a) Find the canonical momenta and invert to express the velocities in canonical form.

(b) Construct the Hamiltonian as a function of canonical variables.