Assignment #5

Reading: Complete Chapter 4 and begin Chapter 5 in *Goldstein*.

Problems: Due by the start of class on Friday, 9/20/19.

(1) Recall that a general rotation matrix can be expressed as a rotation by angle Φ about an axis $\widehat{n}$,

$$A_{ij} = \Big[\delta_{ij} - \widehat{n}_i \widehat{n}_j\Big] \cos(\Phi) + \epsilon_{ijk} \widehat{n}_k \sin(\Phi) + \widehat{n}_i \widehat{n}_j \; .$$

Show that the time derivative of this general form obeys,

$$\Big(A^T \dot{A}\Big)_{ij} = \epsilon_{ijk} \Big[\widehat{n}_k \dot{\Phi} + \dot{\widehat{n}}_k \sin(\Phi) + \epsilon_{k\ell m} \widehat{n}_\ell \dot{\widehat{n}}_m [1 - \cos(\Phi)]\Big] \; .$$

You might find the following identities to be useful,

$$\widehat{n} \cdot \dot{\widehat{n}} = 0 \qquad , \qquad a_i b_j - a_j b_i = \epsilon_{ijk} (\vec{a} \times \vec{b})_k \; .$$

(2) Consider a point particle of mass m which is stationary in a rotating frame with constant angular velocity $\vec{\omega}$.

(a) Show that the centrifugal force on the particle is minus the gradient of the centrifugal potential $V_c = -\frac{1}{2} m \|\vec{\omega} \times \vec{r}\|^2$.

(b) If the particle is on the surface of the Earth it experiences the gravitational potential energy of the Earth, which has a small quadrupole deformation,

$$V_g = -\frac{m M_E G}{r} \left\{1 - J_2 \Big[\tfrac{3}{2} (\widehat{\omega} \cdot \widehat{r})^2 - \tfrac{1}{2}\Big] \Big(\frac{R_E}{r}\Big)^2 \right\} \; .$$

Here $J_2 \simeq .001083$, R_E is the Earth's equatorial radius and M_E is the Earth's mass. Water follows a surface of constant total potential energy which is known as the *geoid*. Solve for the geoid's radius at general polar angle, $R(\theta)$, to first order in the small parameters J_2 and $(\omega^2 R_E^3)/(G M_E)$ and compute the numerical value of the difference of $R(\theta)$ from its largest to its smallest values.

(c) Derive a first-order expression for the acceleration, $g(\theta)$, due to the combined potential, $V_g + V_c$ and compute the numerical value of its largest variation.

(3) Consider a symmetrical top consisting of a solid cone of radius r and height a, joined on its circular face to a solid cylinder of radius r and height b. Both the cone and the cylinder have the same uniform mass density ρ_0.

(a) What is the top's mass?

(b) What is the top's center of mass assuming its axis of rotation is along the $+\widehat{z}$ axis?

(c) What is the moment of inertia tensor I_{ij} taking the apex of the cone to be the origin?

(d) What are the eigenvalues and eigenvectors of I_{ij}?