PHY 6246 September 9, 2019

Assignment #4

Reading: Begin Chapter 4 in *Goldstein*.

Problems: Due by the start of class on Monday, 9/16/19.

(1) Consider a particle of mass m which slides, without rolling and without friction, along the paraboloid of rotation $z = k(x^2 + y^2)$ in the Earth's uniform gravitational field.

(a) Express the Lagrangian using generalized coordinates.

(b) Reduce the system to a 1-dimensional problem.

(c) What is the condition for circular motion?

(d) Suppose the motion is almost circular. What is the period of small oscillations about circular motion?

(2) Recall ϵ_{ijk}, the *Levi-Civita density* which Goldstein defines on page 169. In addition to its use for representing cross products, the Levi-Civita density also facilitates the beautiful representation for a general rotation which Goldstein gives in eqn (4.62') on page 170. This representation involves writing the rotation matrix R_{ij} as the exponential of an infinitesimal rotation $\delta R_{ij} = \Phi \epsilon_{ijk} \widehat{n}_k$,

$$R_{ij} = \exp\Big[\delta R\Big]_{ij} \equiv \delta_{ij} + \sum_{n=1}^{\infty} \tfrac{1}{n!} \Big[\delta R^n\Big]_{ij} \; .$$

In this problem you will derive this representation.

(a) Show that the square of δR is,

$$\Big[\delta R^2\Big]_{ij} = \delta R_{ik} \delta R_{kj} = -\Phi^2 \Big[\delta_{ij} - \widehat{n}_i \widehat{n}_j\Big] \; .$$

(b) Show that the cube of δR is,

$$\Big[\delta R^3\Big]_{ij} = -\Phi^3 \epsilon_{ijk} \widehat{n}_k \; .$$

(c) Use results (a) and (b) to derive explicit relations for $[\delta R^{2n}]_{ij}$ and for $[\delta R^{2n+1}]_{ij}$.

(d) Break the sum over n upon into even and odd powers and use the result of (c) to derive the final result,

$$R_{ij} = \Big[\delta_{ij} - \widehat{n}_i \widehat{n}_j\Big] \cos(\Phi) + \epsilon_{ijk} \widehat{n}_k \sin(\Phi) + \widehat{n}_i \widehat{n}_j \; .$$

(e) Evaluate each of the three terms in the final representation (part d above) for the special cases of $\widehat{n} = \widehat{x}, \widehat{y}, \widehat{z}$.

(3) The Levi-Civita density is also useful in representing determinants and matrix inverses. To see this, suppose that M_{ij} is a general 3×3 matrix.

(a) Use the rules for making a column expansion of the determinant to show that,

$$\epsilon_{ijk} M_{i\ell} M_{jm} M_{kn} = \det(M) \epsilon_{\ell mn} \ .$$

(b) Use the rules for making a row expansion of the determinant to show that,

$$\epsilon_{\ell mn} M_{i\ell} M_{jm} M_{kn} = \det(M) \epsilon_{ijk} \ .$$

(c) Use (a) or (b), with the $BAC - CAB$ rule, to show,

$$\epsilon_{ijk} \epsilon_{\ell mn} M_{i\ell} M_{jm} M_{kn} = 3! \det(M) \ .$$

(d) Show that the following matrix gives the left and right inverse of M_{ij},

$$\left[M^{-1} \right]_{\ell i} = \frac{\epsilon_{ijk} \epsilon_{\ell mn} M_{jm} M_{kn}}{2! \det(M)} \ .$$