PHY 6246 December 5, 2018

# Exam #2

(1) This problem concerns the electromagnetic field strength tensor and three space-time events (in units of meters) whose $3+1$ decompositions are,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{c}E^n \\ \frac{1}{c}E^m & -\epsilon^{mn\ell}B^\ell \end{pmatrix} \quad , \quad x_1^\mu = \begin{pmatrix} 2 \\ \widehat{x} \end{pmatrix} \quad , \quad x_2^\mu = \begin{pmatrix} 4 \\ \widehat{y} \end{pmatrix} \quad , \quad x_3^\mu = \begin{pmatrix} 6 \\ \widehat{z} \end{pmatrix} \ .$$

(a) $3+1$ decompose $F^{\mu\nu}F_{\mu\nu}$ to express the result in terms of $\vec{E}\cdot\vec{E}$, $\vec{E}\cdot\vec{B}$ and $\vec{B}\cdot\vec{B}$. **(16 points)**

(b) Are events $x_1^\mu$ and $x_3^\mu$ timelike or spacelike separated? **(16 points)**

(c) If you answered "timelike", what is the boost which makes the two events occur at the same space point, and what is their temporal difference in that frame? If you answered "spacelike", what is the boost which makes the events simulataneous, and what is their spatial separation in that frame? **(16 points)**

(d) Consider a particle which moves in a straight line from $x_1^\mu$ to $x_2^\mu$. Suppose we boost to the frame of another particle which moves in a straight line from $x_2^\mu$ to $x_3^\mu$. What is the 3-velocity $\vec{u}'$ of the first particle in the rest frame of the second particle? **(16 points)**

(e) Consider two particles which have masses $m_1$ and $m_2$. The first particle has the velocity to go from $x_1^\mu$ to $x_2^\mu$, and the second particle has the velocity to go from $x_2^\mu$ to $x_3^\mu$. What is the inner product $p_1^\mu p_2^\nu g_{\mu\nu}$ of the particle's 4-momenta? **(16 points)**

(2) The first three parts of this problem concerns a system of two coordinates $q_1$ and $q_2$, with their associated canonical momenta $p_1$ and $p_2$. Consider the point transformation to new coorindates $Q_1 = q_1^3$ and $Q_2 = q_1 + q_2$.

(a) What are the most general functions $P_1(q_1, q_2, p_1, p_2)$ and $P_2(q_1, q_2, p_1, p_2)$ such that the new coordinates are canonical? **(17 points)**

(b) Apply your result to the Hamiltonian,

$$H = \frac{a}{2}\Bigl(\frac{p_1 - p_2}{3q_1^2}\Bigr)^2 + bp_2 + c(q_1 + q_2)^2 \ ,$$

where $a$, $b$ and $c$ are constants, to find a system in which $Q_1$ and $Q_2$ are ignorable. **(17 points)**

(c) Use your result for (b) to find the full initial value solutions for $q_1(t)$, $q_2(t)$, $p_1(t)$ and $p_2(t)$. **(17 points)**

(d) Now consider the completely different system of a particle of mass $m$ which is constrained to move in the uniform gravitational field of the Earth (with the $y$ axis vertical) on the curve,

$$x(\lambda) = \ell\Bigl(2\lambda + \sin(2\lambda)\Bigr) \qquad , \qquad y(\lambda) = \ell\Bigl(1 - \cos(2\lambda)\Bigr) \ .$$

What is the Hamiltonian of this system? **(17 points)**

(e) Assuming the system system of part (d) undergoes libration, use action-angle variables to obtain an exact integral expression (which you do not need to evaluate) for the frequency. **(17 points)**

(3) Electromagnetism can be coupled to a massless complex scalar $\varphi(t, \vec{x})$ using the Lagrange density,

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \Big(\partial_\mu + ieA_\mu\Big)\varphi\Big(\partial^\mu - ieA^\mu\Big)\varphi^* \; .$$

(a) Prove that the Lagrangian is invariant under the gauge transformation,

$$A'(t, \vec{x}) = A_\mu(t, \vec{x}) - \partial_\mu \theta(t, \vec{x}) \qquad , \qquad \varphi'(t, \vec{x}) = \exp[ie\theta(t, \vec{x})] \times \varphi(t, \vec{x}) \; .$$

**(16 points)**

(b) What are the equations of motion for the vector potential $A_\mu(t, \vec{x})$ and for the scalar? **(16 points)**

(c) In temporal gauge ($A_0(t, \vec{x}) = 0$) what is the Hamiltonian and the initial value constraint? **(16 points)**

(d) What are the free field solutions for $A_i(t, \vec{x})$ and $\varphi(t, \vec{x})$? Make sure you specify any constraints the initial values of the fields must obey. **(16 points)**

(e) Use Noether's theorem to construct the conserved current associated with the gauge transformation of part (a) for the special case of $\theta(t, \vec{x})$ being a constant in space and time. **(16 points)**