PHY 6246 December 4, 2019

Exam #2 Solutions

(1a) First compute the so-called "dual" field strength tensor,

$$c\widetilde{F}^{\rho\sigma} \equiv \epsilon^{\rho\sigma\mu\nu} cF_{\mu\nu} = \begin{pmatrix} 0 & \epsilon^{smn} F_{mn} \\ -\epsilon^{rmn} F_{mn} & 2\epsilon^{rsn} F_{0n} \end{pmatrix} = \begin{pmatrix} 0 & -2cB^s \\ +2cB^r & 2\epsilon^{rsn} E^n \end{pmatrix} .$$

Now $3+1$ decompose the contraction of this into $F_{\rho\sigma}$,

$$F_{\rho\sigma} \widetilde{F}^{\rho\sigma} = 2F_{0i} \widetilde{F}^{0i} + F_{ij} \widetilde{F}^{ij} = -\tfrac{8}{c} \vec{E} \cdot \vec{B} .$$

(1b) The invariant interval is positive,

$$(x_2^\mu - x_1^\mu)(x_2^\nu - x_1^\nu)\eta_{\mu\nu} = -1 + \|\widehat{x} + \widehat{y}\|^2 = +1 .$$

Hence the events are spacelike separated.

(1c) The primed time difference is $\Delta t' = \gamma(1 - \vec{\beta} \cdot (\widehat{x} + \widehat{y}))$. This will vanish if we take $\vec{\beta} = \frac{1}{2}(\widehat{x} + \widehat{y})$. Of course the primed spatial separation is $+1$.

(1d) The particle's velocity is $\vec{u} = \frac{1}{3}\widehat{y}c$. We are boosting to the rest frame of a particle with velocity $\vec{v} = -\frac{1}{2}\widehat{x}c$, which means $\beta = \frac{1}{2}$ and $\gamma = \frac{2}{\sqrt{3}}$. Because $\vec{u}$ and $\vec{v}$ are orthogonal we have $\vec{u}_\perp = \vec{u}$ and $u_\parallel = 0$. Hence the particle's velocity in the primed frame is,

$$\vec{u}' = \frac{1}{1 - \frac{\vec{u}\cdot\vec{v}}{c^2}} \Big[\sqrt{1-\beta^2}\, \vec{u}_\perp + \vec{u}_\parallel - \vec{v}\Big] = \Big(\tfrac{1}{2}\widehat{x} + \tfrac{1}{\sqrt{12}}\widehat{y}\Big) c .$$

(1e) The two 4-momenta are,

$$p_a^\mu = m_a \gamma_u \begin{pmatrix} c \\ \vec{u} \end{pmatrix} \quad , \quad p_b^\mu = m_b \gamma_v \begin{pmatrix} c \\ \vec{v} \end{pmatrix} .$$

Hence their inner product is,

$$p_a^\mu p_b^\nu \eta_{\mu\nu} = m_a m_b \gamma_u \gamma_v (-c^2 + \vec{u} \cdot \vec{v}) = -\sqrt{\tfrac{3}{2}} m_a m_b c^2 .$$

(2a) For the transformation to be canonical we need $\{Q, P\} = 1$. This means,

$$\Big\{\tfrac{1}{q}, P(q,p)\Big\} = -\frac{1}{q^2}\frac{\partial P}{\partial p} = 1 \qquad \Longrightarrow P(q,p) = -q^2 p + F(q) ,$$

where $F(q)$ is an arbitrary function of q.

(2b) Take $F(q) = 0$ to get $H = \frac{1}{2}P^2 + \frac{1}{2}Q^2$ which implies,

$$Q(t) = Q_0 \cos(t) + P_0 \sin(t) = \tfrac{1}{q_0} \cos(t) - q_0^2 p_0 \sin(t) \; ,$$
$$P(t) = P_0 \cos(t) - Q_0 \sin(t) = -q_0^2 p_0 \cos(t) - \tfrac{1}{q_0} \sin(t) \; .$$

Hence the general initial value solution is,

$$q(t) = \frac{q_0}{\cos(t) - q_0^3 p_0 \sin(t)} \quad , \quad p(t) = \Big[p_0 \cos(t) + \tfrac{1}{q_0^3} \sin(t)\Big]\Big[\cos(t) - q_0^3 p_0 \sin(t)\Big]^2 \; .$$

(2c) The Hamiltonian is $H = \frac{p^2}{2m\ell^2} + mg\ell[1 - \cos(\theta)]$. If the energy is E then the maximum angle and the action variable are,

$$\theta_m = \cos^{-1}\Big(1 - \tfrac{E}{mg\ell}\Big) \quad \Longrightarrow \quad J(E) = 2 \int_{\theta_m}^{-\theta_m} d\theta \sqrt{2m\ell^2 E - 2m^2 g \ell^3 [1 - \cos(\theta)]} \; .$$

Hence the frequency is,

$$f(E) = \frac{1}{J'(E)} = \sqrt{\frac{g}{2\ell}} \Bigg[\int_{-\theta_m}^{\theta_m} \frac{d\theta}{\sqrt{\cos(\theta) - \cos(\theta_m)}} \Bigg]^{-1} \; .$$

(2d) Just make the small angle expansion,

$$\cos(\theta) - \cos(\theta_m) = \tfrac{1}{2}\Big(\theta_m^2 - \theta^2\Big)\Big\{1 - \tfrac{1}{12}\Big(\theta_m^2 + \theta^2\Big) + \dots\Big\} \; .$$

Now substitute, expand and make the change of variable $\theta = \theta_m \sin(\alpha)$ to find,

$$f(E) = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}\Big\{1 - \tfrac{1}{16}\theta_m^2 + O(\theta_m^4)\Big\} = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}\Big\{1 - \frac{E}{8mg\ell} + \dots\Big\} \; .$$

(2e) The exact equation of motion is $\ddot{\theta} = -\frac{g}{\ell}\sin(\theta)$. If we define $\omega^2 \equiv \frac{g}{\ell}$ and expand in small angles, $\theta(t) = \theta^{(0)}(t) + \theta^{(1)}(t) + \dots$, then the 0th and 1st order equations are,

$$\ddot{\theta}^{(0)} + \omega^2 \theta^{(0)} = 0 \qquad , \qquad \ddot{\theta}^{(1)} + \omega^2 \theta^{(1)} = \tfrac{1}{6}\omega^2 \Big[\theta^{(0)}\Big]^3 \; .$$

The 0th order solution is,

$$\theta^{(0}(t) = \theta_0 \cos(\omega t) + \frac{\dot{\theta}_0}{\omega} \sin(\omega t) \equiv A e^{-i\omega t} + A^* e^{i\omega t} \; .$$

The 1st order solution is,

$$\theta^{(1)}(t) = \tfrac{1}{6}\omega^2 \int_0^t dt' \frac{\sin[\omega(t-t')]}{\omega} \Big[A e^{-i\omega t'} + A^* e^{i\omega t'}\Big]^3 \; ,$$
$$= -\tfrac{1}{4} A A^* t \times \dot{\theta}^{(0)}(t) + \Delta A e^{-i\omega t} + \Delta A^* e^{i\omega t} - \tfrac{1}{48}\Big[A^3 e^{-3i\omega t} + A^{*3} e^{3i\omega t}\Big] \; ,$$

where the amplitude shift is $\Delta A = \frac{1}{24}A^3 + \frac{1}{8}A^2 A^* - \frac{1}{8}AA^{*2} - \frac{1}{48}A^{*3}$. Note that the fractional frequency shift of $-\frac{1}{4}AA^*$ agrees with part (d).

(3a) Recall that Fourier transforming carries $\vec{\nabla}$ to $+i\vec{k}$. Hence Maxwell's equations become,

$$i\vec{k}\cdot\widetilde{\vec{E}} = \frac{\widetilde{\rho}}{\epsilon_0} \quad , \quad i\vec{k}\cdot\widetilde{\vec{B}} = 0 \; ,$$
$$i\vec{k}\times\widetilde{\vec{B}} - \frac{1}{c^2}\dot{\widetilde{\vec{E}}} = \mu_0\widetilde{\vec{J}} \quad , \quad i\vec{k}\times\widetilde{\vec{E}} + \dot{\widetilde{\vec{B}}} = 0 \; .$$

(3b) The decoupled equations are,

$$(\partial_t^2 + c^2k^2)\widetilde{\vec{E}} = -c^2\Big[\mu_0\dot{\widetilde{\vec{J}}} + \frac{i\vec{k}}{\epsilon_0}\widetilde{\rho}\Big] \quad , \quad (\partial_t^2 + c^2k^2)\widetilde{\vec{B}} = i\mu_0 c^2\vec{k}\times\widetilde{\vec{J}} \; .$$

(3c) The complete initial value solutions in Fourier space are,

$$\begin{aligned}\widetilde{\vec{E}}(t,\vec{k}) &= \widetilde{\vec{E}}_0(\vec{k})\cos(ckt) + \frac{c}{k}\Big[i\vec{k}\times\widetilde{\vec{B}}_0(\vec{k}) - \mu_0\widetilde{\vec{J}}_0(\vec{k})\Big]\sin(ckt) \\ &\qquad - \frac{c}{k}\int_0^t dt'\,\sin[ck(t-t')]\Big[\mu_0\dot{\widetilde{\vec{J}}}(t',\vec{k}) + \frac{i\vec{k}}{\epsilon_0}\widetilde{\rho}(t',\vec{k})\Big] \; , \\ \widetilde{\vec{B}}(t,\vec{k}) &= \widetilde{\vec{B}}_0(\vec{k})\cos(ckt) - \frac{i}{ck}\vec{k}\times\widetilde{\vec{E}}_0(\vec{k})\sin(ckt) \\ &\qquad + \frac{i\mu_0 c}{k}\int_0^t dt'\,\sin[ck(t-t')]\vec{k}\times\widetilde{\vec{J}}(t',\vec{k})\end{aligned} \; .$$

The position-space results are obtained by inverse-Fourier transforming,

$$\vec{E}(t,\vec{x}) = \int\frac{d^3k}{(2\pi)^3}\,e^{i\vec{k}\cdot\vec{x}}\widetilde{\vec{E}}(t,\vec{k}) \quad , \quad \vec{B}(t,\vec{x}) = \int\frac{d^3k}{(2\pi)^3}\,e^{i\vec{k}\cdot\vec{x}}\widetilde{\vec{B}}(t,\vec{k}) \; .$$

(3d) It's best to do this in Fourier space,

$$\widetilde{\vec{E}}(t,\vec{k}) = -i\vec{k}\widetilde{\Phi}(t,\vec{k}) - \dot{\widetilde{\vec{A}}}(t,\vec{k}) \quad , \quad \widetilde{\vec{B}}(t,\vec{k}) = i\vec{k}\times\widetilde{\vec{A}}(t,\vec{k}) \; .$$

Then the Coulomb gauge condition implies,

$$\widetilde{\Phi}(t,\vec{k}) = \frac{1}{\epsilon_0 k^2}\widetilde{\rho}(t,\vec{k}) \quad , \quad \widetilde{\vec{A}}(t,\vec{k}) = \frac{i}{k^2}\vec{k}\times\widetilde{\vec{B}}(t,\vec{k}) \; .$$

Inverse-Fourier transforming gives the potentials. Note that $\Phi(t,\vec{x})$ depends only on the current density; all the photon degrees of freedom reside in $\vec{A}(t,\vec{x})$.

(3e) The variation of the field $A_{b\nu}$ with respect to spacetime constant parameter θ_a is $\Delta_a A_{b\nu} = gf_{abc}A_{c\nu}$. Because the symmetry is internal we also have $\mathcal{J}_a^\mu = 0$. Hence the Noether current and its associated charge are:

$$J_a^\mu(t,\vec{x}) \equiv \frac{\partial\mathcal{L}}{\partial\partial_\mu A_{b\nu}}\Delta_a A_{b\nu} - \mathcal{J}_a^\mu = gf_{abc}A_{b\nu}(t,\vec{x})F_c^{\mu\nu}(t,\vec{x}) \Longrightarrow Q_a(t) = \int d^3x J_a^0(t,\vec{x}) \; .$$