PHY 6246 December 4, 2019

Exam #2

(1) This problem concerns the electromagnetic field strength tensor and three space-time events (in units of meters) whose 3 + 1 decompositions are,

$$cF_{\mu\nu} = \begin{pmatrix} 0 & E^n \\ -E^m & -\epsilon^{mn\ell} cB^\ell \end{pmatrix} \quad , \quad x_1^\mu = \begin{pmatrix} 0 \\ \vec{0} \end{pmatrix} \quad , \quad x_2^\mu = \begin{pmatrix} 1 \\ \widehat{x} + \widehat{y} \end{pmatrix} \quad , \quad x_3^\mu = \begin{pmatrix} 3 \\ \widehat{y} \end{pmatrix} \ .$$

(a) 3 + 1 decompose $\epsilon^{\rho\sigma\mu\nu} F_{\rho\sigma} F_{\mu\nu}$ to express the result in terms of $\vec{E} \cdot \vec{E}$, $\vec{E} \cdot \vec{B}$ and $\vec{B} \cdot \vec{B}$. **(16 points)**

(b) Are events x_1^μ and x_2^μ timelike or spacelike separated? **(16 points)**

(c) If you answered "timelike", what is the boost which makes the two events occur at the same space point, and what is their temporal difference in that frame? If you answered "spacelike", what is the boost which makes the events simultaneous, and what is their spatial separation in that frame? **(16 points)**

(d) Consider a particle which moves in a straight line from x_1^μ to x_3^μ. Suppose we boost to the frame of another particle which moves in a straight line from x_2^μ to x_3^μ. What is the 3-velocity $\vec{u}'$ of the first particle in the rest frame of the second particle? **(16 points)**

(e) Consider two particles which have masses m_a and m_b. Particle a has the velocity to go from x_1^μ to x_3^μ, and particle b has the velocity to go from x_2^μ to x_3^μ. What is the inner product $p_a^\mu p_b^\nu \eta_{\mu\nu}$ of the two particle's 4-momenta? **(16 points)**

(2) The first two parts of this problem concern a one dimensional problem whose canonically conjugate coordinates are q and p.

(a) Suppose we make the point transformation to a new coordinate $Q = \frac{1}{q}$. What is the most general conjugate momentum $P(q, p)$? **(17 points)**

(b) Suppose the original Hamiltonian is $H = \frac{1}{2q^2} + \frac{1}{2} q^4 p^2$. Use the transformation of part (a) to derive the general initial value solution for $q(t)$ and $p(t)$. **(17 points)**

(c) Recall the nonlinear pendulum of mass m and bob length ℓ which is characterized the angle $\theta(t)$ it makes with the vertical. The Lagrangian is,

$$L = \tfrac{1}{2} m \ell^2 \dot{\theta}^2 - mg\ell [1 - \cos(\theta)] \ .$$

Use action-angle variables to find an exact integral expression for the libration frequency $f(E)$ as a function of the energy E. **(17 points)**

(d) Compute the small angle result $f(0)$, and the first order (order $\frac{E}{mg\ell}$) correction. **(17 points)**

(e) Use perturbation theory to solve for $\theta(t)$ in the small angle limit, and its first correction about that limit. **(17 points)**

(3) In parts (a-d) we will assume that the string theorists are too preoccupied to meddle with electrodynamics, so that Maxwell's equations are still valid,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad , \quad \vec{\nabla} \cdot \vec{B} = 0 \; ,$$
$$\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \dot{\vec{E}} = \mu_0 \vec{J} \quad , \quad \vec{\nabla} \times \vec{E} + \dot{\vec{B}} = 0 \; .$$

In this problem you are to find the **general** solutions for $\vec{E}(t, \vec{x})$ and $\vec{B}(t, \vec{x})$ assuming that $\rho(t, \vec{x})$ and $\vec{J}(t, \vec{x})$ are fixed, known but arbitrary functions of space and time.

(a) Spatially Fourier transform Maxwell's equations and express your result in terms of $\widetilde{\vec{E}}(t, \vec{k})$, $\widetilde{\vec{B}}(t, \vec{k})$, $\widetilde{\rho}(t, \vec{k})$ and $\widetilde{\vec{J}}(t, \vec{k})$. **(16 points)**

(b) Eliminate the coupling between $\widetilde{\vec{E}}$ and $\widetilde{\vec{B}}$ by taking a time derivative of each of the bottom equations and then using the other equations. **(16 points)**

(c) Give integral expressions for $\vec{E}(t, \vec{x})$ and $\vec{B}(t, \vec{x})$ in terms of their initial values and the charge and current densities. **(16 points)**

(d) Recall that the electric and magnetic field can be expressed using a scalar potential $\Phi(t, \vec{x})$ and a vector potential $\vec{A}(t, \vec{x})$,

$$\vec{E}(t, \vec{x}) = -\vec{\nabla}\Phi(t, \vec{x}) - \dot{\vec{A}}(t, \vec{x}) \quad , \quad \vec{B}(t, \vec{x}) = \vec{\nabla} \times \vec{A}(t, \vec{x})$$

Assume the Coulomb gauge condition $\vec{\nabla} \cdot \vec{A}(t, \vec{x}) = 0$ and use your solutions from part (c) to find the general solutions for $\Phi(t, \vec{x})$ and $\vec{A}(t, \vec{x})$. **(16 points)**

(e) Now make the further assumption that the string theorists have not changed either the Yang-Mills field strength or Lagrangian density,

$$F_{a\mu\nu} = \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} - g f_{abc} A_{b\mu} A_{c\nu} \quad , \quad \mathcal{L} = -\tfrac{1}{4} F_{a\mu\nu} F_a{}^{\mu\nu} \; .$$

Recall that, for arbitrary gauge parameter $\theta_a(t, \vec{x})$, this theory is invariant under the internal symmetry transformation,

$$A'_{a\mu}(t, \vec{x}) = A_{a\mu}(t, \vec{x}) - \partial_\mu \theta_a(t, \vec{x}) + g f_{abc} A_{b\mu}(t, \vec{x}) \theta_c(t, \vec{x}) \; .$$

Suppose $\theta_a(t, \vec{x})$ is constant in space and time, and use Noether's theorem to construct the conserved currents (one for each value of a) $J_a{}^\mu(t, \vec{x})$ and the associated charges $Q_a(t)$. **(16 points)**