Exam #1 Solutions

(1a) * $M = (m_1 + m_2)$ & $m = \frac{m_1 m_2}{m_1 + m_2}$

where $\ell \equiv L_1/m$ & $e \equiv E/m$

(1b) * $L_1 = m r^2 \dot{\phi}$

* $E = \frac{1}{2} m \dot{r}^2 + \frac{L_1^2}{2 m r^2} - \frac{G M m}{r}\left(1 + \frac{3 G M}{r c^2}\right)$ $\Big\} \rightarrow$ $\dot{\phi} = \frac{\ell}{r^2}$, $\dot{r} = \pm\sqrt{2e - \frac{\ell^2}{r^2} + \frac{2 G M}{r}\left(1 + \frac{3 G M}{c^2 r}\right)}$

* Defining $\alpha \equiv 1 - 6\left(\frac{G M}{c \ell}\right)^2$ gives $\dot{r} = \pm\sqrt{2e + \frac{2 G M}{r} - \frac{\alpha \ell^2}{r^2}}$ (similar to Kepler $\rightarrow$ known)

(1c) * $\dot{r} = 0 \rightarrow \frac{1}{r_\pm} = \frac{G M}{\alpha \ell^2}\left[1 \pm \sqrt{1 + \frac{2 e \alpha \ell^2}{G^2 M^2}}\right]$ $\rightarrow$ define $\epsilon \equiv \sqrt{1 + \frac{2 e \alpha \ell^2}{G^2 M^2}}$

(1d) * $r' = \frac{\dot{r}}{\dot{\phi}} = \pm \frac{r^2}{\ell^2}\sqrt{2e + \frac{2 G M}{r} - \frac{\alpha \ell^2}{r^2}}$ $\rightarrow$ $u' = \mp\sqrt{\alpha}\sqrt{-u^2 + \frac{2 G M}{\alpha \ell^2} u + \frac{2e}{\alpha \ell^2}}$

* using the integral (with $a = \frac{G M}{\alpha \ell^2}$ & $b = \frac{-2e}{\alpha \ell^2}$ $\rightarrow \sqrt{a^2 - b} = \epsilon * \frac{G M}{\alpha \ell^2}$) gives

$u(\phi) = \frac{G M}{\alpha \ell^2}\left[1 + \epsilon \sin\left[\sqrt{\alpha}(\phi - \phi_0)\right]\right]$ $\rightarrow$ $r(\phi) = \frac{\alpha \ell^2 / G M}{1 + \epsilon \sin\left[\sqrt{\alpha}(\phi - \phi_0)\right]}$

* NB this is not quite Kepler because $\alpha \neq 1$

(1e) * we can actually recognize $\Delta\phi = 2\pi\left[\frac{1}{\sqrt{\alpha}} - 1\right] \simeq 2\pi * 3\left(\frac{G M}{c \ell}\right)^2$ from (1d)

* However, the perturbative technique also works

* $e = \frac{\ell^2}{2} u'^2 + \frac{1}{2}\alpha \ell^2 u^2 - G M u \rightarrow \ell^2 u'' = G M - \alpha \ell^2 u \rightarrow$ circle for $\left\{ u_0 = \frac{G M}{\alpha \ell^2},\ e = -G^2 M^2 / \alpha \ell^2 \right\}$

* $u = u_0 + \Delta u \rightarrow \ell^2 \Delta u'' = -\alpha \ell^2 \Delta u \rightarrow \Delta\phi = 2\pi\left[\frac{1}{\sqrt{\alpha}} - 1\right]$

(2a) * $M = 2m$ & $\vec{R}_1 = \frac{L}{2}\hat{x}$ + $\vec{R}_2 = \frac{L}{2}\hat{y}$ $\rightarrow$ $\vec{R} = \frac{L}{4}(\hat{x} + \hat{y})$

(2b) * I_{ij} for the x-axis bar is its COM value + shift by $\frac{L}{4}(-\hat{x} + \hat{y})$ to $\vec{R}$

$$\rightarrow m L_1^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & \frac{1}{12} \end{pmatrix} + m L_1^2 \begin{pmatrix} \frac{1}{16} & \frac{1}{16} & 0 \\ \frac{1}{16} & \frac{1}{16} & 0 \\ 0 & 0 & \frac{1}{8} \end{pmatrix} = m L_1^2 \begin{pmatrix} \frac{1}{16} & \frac{1}{16} & 0 \\ \frac{1}{16} & \frac{7}{48} & 0 \\ 0 & 0 & \frac{5}{24} \end{pmatrix}$$

* I_{ij} for the y-axis bar is its COM value + shift by $\frac{L}{4}(\hat{x} - \hat{y})$ to $\vec{R}$

$$\rightarrow m L_1^2 \begin{pmatrix} \frac{1}{12} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{12} \end{pmatrix} + m L_1^2 \begin{pmatrix} \frac{1}{16} & \frac{1}{16} & 0 \\ \frac{1}{16} & \frac{1}{16} & 0 \\ 0 & 0 & \frac{1}{8} \end{pmatrix} = m L_1^2 \begin{pmatrix} \frac{7}{48} & \frac{1}{16} & 0 \\ \frac{1}{16} & \frac{1}{16} & 0 \\ 0 & 0 & \frac{5}{24} \end{pmatrix}$$

$\therefore$ total I_{ij} about $\vec{R}$ is $m L_1^2 \begin{pmatrix} \frac{5}{24} & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{5}{24} & 0 \\ 0 & 0 & \frac{5}{12} \end{pmatrix}$

(2c) * $\det(I - \lambda) = 0 \rightarrow \lambda_1 = \frac{1}{12} m L_1^2$, $\lambda_2 = \frac{1}{3} m L_1^2$ & $\lambda_3 = \frac{5}{12} m L_1^2$

* principal axes are $\vec{a}_1 = \frac{1}{\sqrt{2}}(\hat{x} - \hat{y})$, $\vec{a}_2 = \frac{1}{\sqrt{2}}(\hat{x} + \hat{y})$ & $\vec{a}_3 = \hat{z}$

(2d) * $\dot{\omega}_1 = -\left(\frac{I_{33} - I_{22}}{I_{11}}\right)\omega * \omega_2 = -\omega \omega_2$

* $\dot{\omega}_2 = +\left(\frac{I_{33} - I_{11}}{I_{22}}\right)\omega * \omega_1 = +\omega \omega_1$ $\Big\} \rightarrow \left\{ \begin{array}{l} \omega_1(t) = \omega_{10}\cos(\omega t) - \omega_{20}\sin(\omega t) \\ \omega_2(t) = \omega_{20}\cos(\omega t) + \omega_{10}\sin(\omega t) \end{array} \right\} \rightarrow$ rate is ω

(2e) * $L_{1i} = I_{ij}\omega_j$ (in principal axes coords $\vec{a}_1$, $\vec{a}_2$ & $\vec{a}_3$)

$\rightarrow L_{11}(t) = \frac{1}{12} m L_1^2 \omega_1(t)$, $L_{12}(t) = \frac{1}{3} m L_1^2 \omega_2(t)$ & $L_{13} = \frac{5}{12} m L_1^2 \omega$

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(3a) $* \; L = \frac{1}{2}m\dot{x}_1^2 + m\dot{x}_2^2 + \frac{3}{2}m\dot{x}_3^2 - \frac{1}{2}k(x_2 - x_1 - a)^2 - \frac{3}{2}k(x_3 - x_2 - b)^2$

(3b) $$\left.\begin{aligned} * \; m\ddot{x}_1 &= k(x_2 - x_1 - a) \\ * \; 2m\ddot{x}_2 &= -k(x_2 - x_1 - a) + 3k(x_3 - x_2 - b) \\ * \; 3m\ddot{x}_3 &= -3k(x_3 - x_2 - b) \end{aligned}\right\} + \ddot{x}_i = 0 \rightarrow \left\{\begin{aligned} \overline{x}_1 &= \overline{X} \\ \overline{x}_2 &= \overline{X} + a \\ \overline{x}_3 &= \overline{X} + a + b \end{aligned}\right\}$$

(3c) $$x_i = \overline{x}_i + \eta_i \rightarrow \left\{\begin{aligned} m\ddot{\eta}_1 &= k(\eta_2 - \eta_1) \\ 2m\ddot{\eta}_2 &= -k(\eta_2 - \eta_1) + 3k(\eta_3 - \eta_2) \\ 3m\ddot{\eta}_3 &= -3k(\eta_3 - \eta_2) \end{aligned}\right\} \rightarrow \Omega^2 = -\frac{k}{m}\begin{pmatrix} 1 & -1 & 0 \\ -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & -1 & 1 \end{pmatrix}$$

$$* \det\begin{bmatrix} 1-\lambda & -1 & 0 \\ -\frac{1}{2} & 2-\lambda & -\frac{3}{2} \\ 0 & -1 & 1-\lambda \end{bmatrix} = -\lambda(1-\lambda)(3-\lambda) \rightarrow \omega^2 = 0, \; \frac{k}{m} \;\&\; 3\frac{k}{m}$$

(3d) $$\begin{pmatrix} 1 & -1 & 0 \\ -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y \\ -\frac{1}{2}x + 2y - \frac{3}{2}z \\ -y + z \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \lambda = 0 \text{ has } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; \lambda = 1 \text{ has } \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{3} \end{pmatrix} \;\&\; \lambda = 3 \text{ has } \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

(3e) * This one is tedious!

* First subtract the equilibrium $\rightarrow \Delta\vec{x}_0 = \begin{pmatrix} L \\ 2L - a \\ 3L - a - b \end{pmatrix}$

* Now express $\Delta\vec{x}_0$ as a sum of the eigenvectors $\vec{u}_i$

$$\rightarrow \Delta\vec{x}_0 = \sum_{i,j=1}^{3} \vec{u}_i M^{-1}_{ij} \vec{u}_j \cdot \Delta\vec{x}_0 \text{ where } M_{ij} \equiv \vec{u}_i \cdot \vec{u}_j = \begin{pmatrix} 3 & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{10}{9} & \frac{2}{3} \\ 0 & \frac{2}{3} & 6 \end{pmatrix}$$

$$\rightarrow M^{-1}_{ij} = \frac{1}{16}\begin{pmatrix} \frac{56}{9} & -4 & \frac{4}{9} \\ -4 & 18 & -2 \\ \frac{4}{9} & -2 & \frac{26}{9} \end{pmatrix}$$

$\& \; \vec{u}_1 \cdot \Delta\vec{x}_0 = 6L - 2a - b$
$\vec{u}_2 \cdot \Delta\vec{x}_0 = \frac{1}{3}(a + b) \;\; \& \;\; \vec{u}_3 = a - b$

* Finally, evolve each $\vec{u}_i$ by $\cos(\omega_i t)$

$$\rightarrow \vec{x}(t) = \begin{pmatrix} 0 \\ a \\ a+b \end{pmatrix} + \vec{u}_1\left(\frac{7}{3}L - \frac{5}{6}a - \frac{1}{2}b\right) + \vec{u}_2\left(-\frac{3}{2}L + \frac{3}{4}a + \frac{3}{4}b\right)\cos\left(\sqrt{\frac{k}{m}}t\right) + \vec{u}_3\left(\frac{1}{6}L + \frac{1}{12}a - \frac{1}{4}b\right)\cos\left(\sqrt{\frac{3k}{m}}t\right)$$

$$\therefore \; x_1(t) = \frac{7}{3}L - \frac{5}{6}a - \frac{1}{2}b + \left(-\frac{3}{2}L + \frac{3}{4}a + \frac{3}{4}b\right)\cos\left(\sqrt{\frac{k}{m}}t\right) + \left(\frac{1}{6}L + \frac{1}{12}a - \frac{1}{4}b\right)\cos\left(\sqrt{\frac{3k}{m}}t\right)$$

* NB much of the drudgery can be avoided by taking appropriate correspondence limits for a & b

– Eg taking $a = +L$ & $b = -2L \rightarrow \Delta\vec{x}_0 = L * \vec{u}_3$