(Mid-Term 2) 2019.

1. For the surface

(a) either $E = 0$ $n_1 = 0$
or $= \epsilon_i$ 1

$$Z = \sum_i e^{-(E_i - \mu N_i)/k_B T}$$

$$= 1 + e^{-\beta(\epsilon_i - \mu)}$$

(b) Probability for state $|\epsilon_i\rangle$

$$p_i = \frac{e^{-\beta(\epsilon_i - \mu)}}{Z} = \frac{e^{-\beta(\epsilon_i - \mu)}}{1 + e^{-\beta(\epsilon_i - \mu)}}$$

$$= \frac{1}{e^{\beta(\epsilon_i - \mu)} + 1}$$

(c) $$\frac{n_i}{N} = \frac{1}{e^{\beta(\epsilon_i - \mu)} + 1}$$

(d) $\mu / k_B = 10$ $\mu = 10 k_B$

(i) $\epsilon_i = 100 k_B$ $T = 10K$ $\frac{n_i}{N} = e^{-9} \approx 0$

(ii) $10 k_B$ $100K$ $\frac{n_i}{N} = \frac{1}{e^0 + 1} = \frac{1}{2}$

(iii) $10 k_B$ $0.1K$ $\frac{n_i}{N} = \left(\cancel{1} + e^{\frac{10-10\cancel{0}}{0.1}}\right)^{-1} = \frac{1}{2}$

2. In 3D

$$\mathcal{D}(k)\,dk = \frac{Vk^2}{2\pi}\,\frac{dk}{d\epsilon(k)}\,d\epsilon$$

density of states

(a) $\epsilon = \hbar c k$ $\qquad \mathcal{D}(\epsilon)\,d\epsilon = \dfrac{V\epsilon^2}{2\pi^2\hbar^3 c^3}$

(b)
$$Z_1 = \int_0^\infty e^{-\beta E}\,\mathcal{D}(E)\,dE \quad \text{(for one particle)}$$

$$= \frac{V}{2\pi^2\hbar^3 c^3}\underbrace{\int e^{-\beta\epsilon}\,\beta^2\epsilon^2\,d\epsilon\beta}_{\Gamma(3)=2}\,(k_B T)^3$$

$$= \frac{V\,(k_B T)^3}{\pi^2\hbar^3 c^3}$$

for N particles
$$Z_1^N = \left(\frac{V(k_B T)^3}{\pi^2\hbar^3 c^3}\right)^N \frac{1}{N!}$$

(c)
$$F = -k_B T \ln Z^N$$
$$= k_B T \ln N! - N \ln\left\{\frac{V(k_B T)^3}{\pi^2(\hbar c)^3}\right\}$$

(d)
$$S = -\left(\frac{\partial F}{\partial T}\right)_V = N k_B\left[\left\{\ln\left(\frac{V}{N}\right)\frac{1}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3\right\} + 4\right]$$