Ch. 2 Electrostatics

▸ Work & Energy

↳ Done by the net force on an object.

$$W = \int_{\vec{r}_1}^{\vec{r}_2}{}_{P} \vec{F}_{net} \cdot d\vec{r}$$: in general depends on the path

$$[W] = [E] = N \cdot m = J$$

⊙ Work - Kinetic Energy theorem.

$$W = \Delta E_k = E_k(\vec{r}_2) - E_k(\vec{r}_1)$$: E_k : kinetic energy.

Positive Work done on the object by the net force increases the kinetic energy.

⊙ Potential Energy & Conservation of Mechanical Energy.

If $\vec{F}_{net}$ is a conservative force (vector field),

then $\int_S (\vec{\nabla} \times \vec{F}_{net}) \cdot d\vec{a} = \oint_C \vec{F}_{ext} \cdot d\vec{r} \rightarrow \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_{ext} \cdot d\vec{r}$: path indep.

Therefore one can define a new quantity (pot. energy) uniquely

$$U(\vec{r}) = -\int_{ref}^{\vec{r}} \vec{F}_{net} \cdot d\vec{r}$$

$$\therefore W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_{net} \cdot d\vec{r} = -\left(-\int_{ref}^{\vec{r}_2} \vec{F}_{net} \cdot d\vec{r} + \int_{ref}^{\vec{r}_1} \vec{F}_{net} \cdot d\vec{r}\right)$$

$$= -\left(U(\vec{r}_2) - U(\vec{r}_1)\right)$$

$$= E_k(\vec{r}_2) - E_k(\vec{r}_1).$$

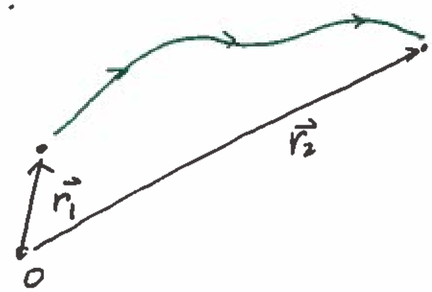
$$\therefore \Delta E_k + \Delta U = \Delta(\underbrace{E_k + U}_{\text{conserved quantity}}) = 0$$

- Moving a particle in the presence of a conservative force
  conservative vector field.

◎ To move a particle w/o causing $\Delta E_k \neq 0$. change in K.E.

There must be a force applied ($\vec{f}$).

$$\vec{f} = -\underbrace{\vec{F}_c}_{\text{conservative force}}$$



Then the work done by $\vec{f}$

$$\boxed{W} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{f} \cdot d\vec{r}$$

$$= -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_c \cdot d\vec{r} = -\left\{ \int_{\vec{r}_1}^{\vec{r}_{ref}} \vec{F}_c \cdot d\vec{r} + \int_{\vec{r}_{ref}}^{\vec{r}_2} \vec{F}_c \cdot d\vec{r} \right\} = \int_{ref}^{\vec{r}_1} \vec{F}_c \cdot d\vec{r} - \int_{ref}^{\vec{r}_2} \vec{F}_c \cdot d\vec{r}$$

$$= U(\vec{r}_2) - U(\vec{r}_1).$$

Amount of work needed to move a particle from $\vec{r}_1$ to $\vec{r}_2$ in the presence of a conservative force. = Change in P.E.

◎ For a charge Q in the presence of $\vec{E}$:

$$\underbrace{U(\vec{r}_2) - U(\vec{r}_1)}_{\text{Change in E. Pot. Energy}} = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_Q \cdot d\vec{r} = -\int_{\vec{r}_1}^{\vec{r}_2} Q \vec{E} \cdot d\vec{r}$$

$$= -Q \int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{r} = Q\,(\underbrace{V(\vec{r}_2) - V(\vec{r}_1)}_{\text{change in E. Potential}}).$$

$[U]$ = Joule.

$[V] = J/C = V$ (volt).

◎ Set the reference pt at ∞, then

$W = Q\,[V(\vec{r}) - V(\infty)]$: amount of energy (work) needed to move charge Q from the infinite to $\vec{r}$

$V(\infty) = 0$ since $\vec{E} \propto \frac{1}{r^2}$.

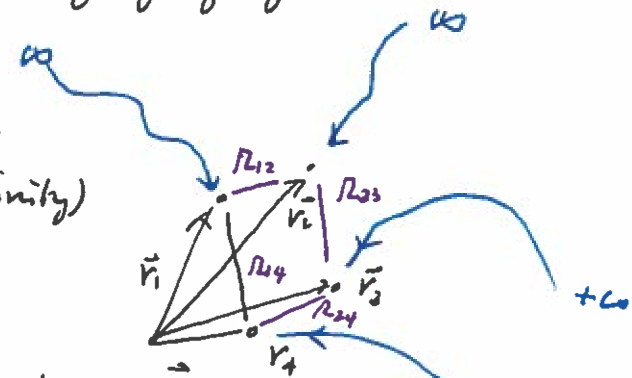
$\therefore\ W = Q\,V(\vec{r})$.

- Energy of a point charge Dist.

Question: How much energy is required to assemble a collection of point charges to a certain configuration?

(i) Bring each charge from infinite to the position one by one.
(ii) Add all energy required.

To begin with, consider 4 charges q_1, q_2, q_3, q_4 at $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4$.



> It does not take any work to bring q_1 to $\vec{r}_1$ because there are no charges (all are at infinity) in the region and $\vec{E} = 0$.

$$W_1 = 0.$$

By placing q_1 at $\vec{r}_1$, now we have $\vec{E}$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{\mathcal{R}_1^2}\hat{\mathcal{R}}_1 \qquad (\vec{\mathcal{R}}_1 = \vec{r} - \vec{r}_1).$$

point charge q at $\vec{r}\,'$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathcal{R}}$$

> Bring q_2 to $\vec{r}_2$.

$$W_2 = q_2 \frac{1}{4\pi\varepsilon_0}\frac{q_1}{\mathcal{R}_{12}} \qquad \vec{\mathcal{R}}_{12} = \vec{r}_1 - \vec{r}_2$$

Now $$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{\mathcal{R}_1^2}\hat{\mathcal{R}}_1 + \frac{q_2}{\mathcal{R}_2^2}\hat{\mathcal{R}}_2\right).$$

> Bring q_3 to $\vec{r}_3$

$$W_3 = q_3\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{\mathcal{R}_{13}} + \frac{q_2}{\mathcal{R}_{23}}\right).$$

Similary

$$W_4 = q_4\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{\mathcal{R}_{14}} + \frac{q_2}{\mathcal{R}_{24}} + \frac{q_3}{\mathcal{R}_{34}}\right).$$

$$\therefore\ W = W_1 + W_2 + W_3 + W_4 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1q_2}{\mathcal{R}_{12}} + \frac{q_1q_3}{\mathcal{R}_{13}} + \frac{q_1q_4}{\mathcal{R}_{14}} + \frac{q_2q_3}{\mathcal{R}_{23}} + \frac{q_2q_4}{\mathcal{R}_{24}} + \frac{q_3q_4}{\mathcal{R}_{34}}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{4}\sum_{j>i}^{4}\frac{q_iq_j}{\mathcal{R}_{ij}} = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{4}\sum_{j\neq i}^{4}\frac{q_iq_j}{\mathcal{R}_{ij}}$$

$$W = \frac{1}{2}\sum_{i=1}^{4} q_i \Big(\underbrace{\sum_{j\neq i}^{4} \frac{1}{4\pi\varepsilon_0}\frac{q_j}{r_{ij}}}_{} \Big)$$

electric potential at $\vec{r}_i$ due to all other charges

position of q_i

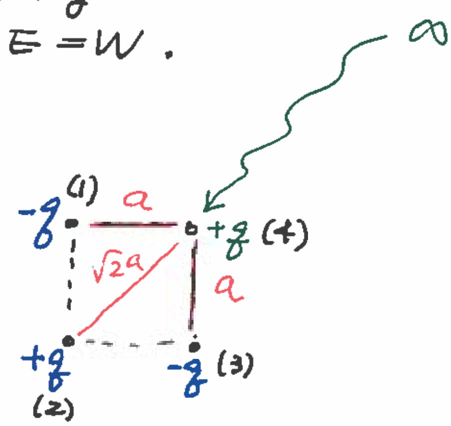
$$= \frac{1}{2}\sum_{i=1}^{4} q_i V(\vec{r}_i).$$

Generalization to n charges

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \Big(\sum_{j\neq i}^{n} \frac{1}{4\pi\varepsilon_0}\frac{q_j}{r_{ij}} \Big).$$

Initially when all charges are at infinity $E=0$.
Now when the charges are assembled. $E=W$.

EX.



(a).

$$W_4 = (+q)\,V(\vec{r}_4)$$

pot. at the position of q_4 from all other charges

$$V(\vec{r}_4) = \frac{1}{4\pi\varepsilon_0}\Big(\underset{(1)}{\frac{-q}{a}} + \underset{(2)}{\frac{q}{\sqrt{2}a}} - \underset{(3)}{\frac{q}{a}} \Big).$$

$$\therefore W_4 = \frac{q^2}{4\pi\varepsilon_0 a}\Big(\frac{1}{\sqrt{2}} - 2\Big) < 0.$$

(b). $$W = \frac{1}{4\pi\varepsilon_0}\Big(\frac{-q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a} - \frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a} \Big)$$

$$= \frac{q^2}{4\pi\varepsilon_0 a}\Big(-4 + \frac{2}{\sqrt{2}}\Big) = \frac{2q^2}{4\pi\varepsilon_0}\Big(-2 + \frac{1}{\sqrt{2}}\Big) < 0.$$

$\boxed{\text{HW}}$ 2.32 & 2.33.

* Energy of a continuous charge dist.

$$W = \frac{1}{2}\int \lambda V dl, \quad \frac{1}{2}\int \sigma V da, \quad \frac{1}{2}\int \rho V d\tau .$$

Line charge   Surface charge   volume charge.

Since $\rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E}$,

$$W = \frac{\varepsilon_0}{2}\int (\vec{\nabla}\cdot\vec{E}) V d\tau$$

$$= \frac{\varepsilon_0}{2}\left[-\int_V \vec{E}\cdot\vec{\nabla}V d\tau + \oint_S \vec{E} V \cdot d\vec{a} \right].$$

See the end of L3. (mistake in L3).

$$\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot\vec{\nabla}f$$

$$\int_V \vec{\nabla}\cdot(f\vec{A})d\tau = \int_V f(\vec{\nabla}\cdot\vec{A})d\tau + \int_V (\vec{A}\cdot\vec{\nabla}f)d\tau.$$

$$\Downarrow$$

$$\oint_S f\vec{A}\cdot d\vec{a} = \int_V f(\vec{\nabla}\cdot\vec{A})d\tau + \int_V \vec{A}\cdot\vec{\nabla}f d\tau$$

$$= \frac{\varepsilon_0}{2}\left[-\int_V \vec{E}\cdot(-\vec{E}) d\tau + \oint_S V\vec{E}\cdot d\vec{a} \right]$$

$$= \frac{\varepsilon_0}{2}\left[\underbrace{\int_V E^2 d\tau}_{\text{volume}} + \underbrace{\oint_S V\vec{E}\cdot\vec{a}}_{\text{surface}} \right]$$

In the original expression $W = \frac{1}{2}\int_V \rho V d\tau$, the integral is done over the volume that contains all charge distribution. Technically it can be integrated for the entire space where $r\to\infty$. The surface integral part

$$\oint_S V\vec{E}\cdot d\vec{a} \sim \lim_{r\to\infty}\int\left(\frac{1}{r}\right)\left(\frac{1}{r^2}\right)\cdot r^2 d\theta d\varphi \sim \lim_{r\to\infty}\int \boxed{\frac{1}{r}} d\theta d\varphi$$

$$\Rightarrow 0$$

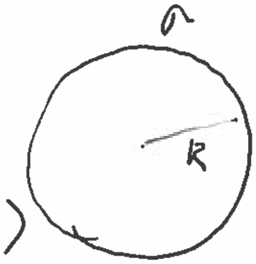
Therefore,

$$W = \frac{1}{2}\varepsilon_0\int_V E^2 d\tau \quad \text{for all space}.$$

E-field in space carries energy!

Therefore $\frac{1}{2}\boxed{\varepsilon_0}E^2$: energy density in free space

Ex



(i) $W = \frac{1}{2}\int_S \sigma V(R)\, da$

$V(R) = \underbrace{\frac{1}{\cancel{4\pi}\epsilon_0}\frac{\cancel{4\pi}R^{\cancel{2}}\sigma}{\cancel{R}}}_{\left(\frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r}\right)} = \frac{R}{\epsilon_0}\sigma \qquad (4\pi R^2\sigma = q)$

$W = \frac{1}{2}\cdot\frac{1}{4\pi\epsilon_0}\frac{\overbrace{4\pi R^2\sigma}^{=q}}{R}\overbrace{\int_S \sigma\, ds}^{=q} = \frac{1}{8\pi\epsilon_0}\frac{q^2}{R}\cdots$

(ii) $W = \frac{\epsilon_0}{2}\int_V E^2\, d\tau. \quad \leftarrow \quad \vec{E}(\vec{r}) = \begin{cases} 0 & \text{for } r<R \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} & \text{for } r>R \end{cases}$

$= \frac{\epsilon_0}{2}\cdot\frac{q^2}{(4\pi\epsilon_0)^2}\int_R^\infty dr\int_0^\pi d\theta\int_0^{2\pi} d\varphi\, \frac{r^2}{r^4}\sin\theta$

$= \frac{\epsilon_0}{2}\frac{q^2}{(4\pi\epsilon_0)^2}\int_R^\infty \frac{1}{r^2}dr\, \underbrace{\boxed{\int_0^\pi\int_0^{2\pi}\sin\theta\, d\theta\, d\varphi}}_{4\pi}$

$= \frac{1}{8\pi\epsilon_0}\cdot\frac{q^2}{R}$

$\boxed{\text{HW}}$ 2.34

$\boxed{\text{HW}}$ 2.36.