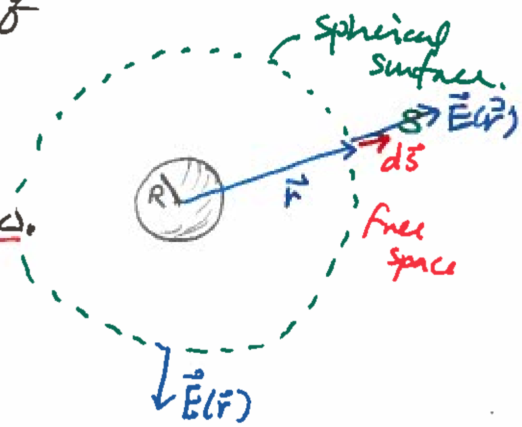
▸ Applications of Gauss's law

Problems with high symm. (spherical, cylindrical, infinite, ⋯)

EX $\vec{E}(\vec{r})$ for $r>R$ from a $\underbrace{\text{uniformly charged}}_{q}$ solid sphere of R

$$\oint_S \vec{E}\cdot d\vec{s} = \frac{Q_{enc}}{\varepsilon_0} = q/\varepsilon_0$$



*By symmetry, $\vec{E}$ is in $\hat{r}$ and isotropic.
and $d\vec{s} = ds\,\hat{r}$

$$\therefore \oint_S \vec{E}\cdot d\vec{s} = \oint_S |E(r)|\,ds\,\underbrace{(\hat{r}\cdot\hat{r})}_{=1}$$

$$= 4\pi r^2 |E(r)| = q/\varepsilon_0$$

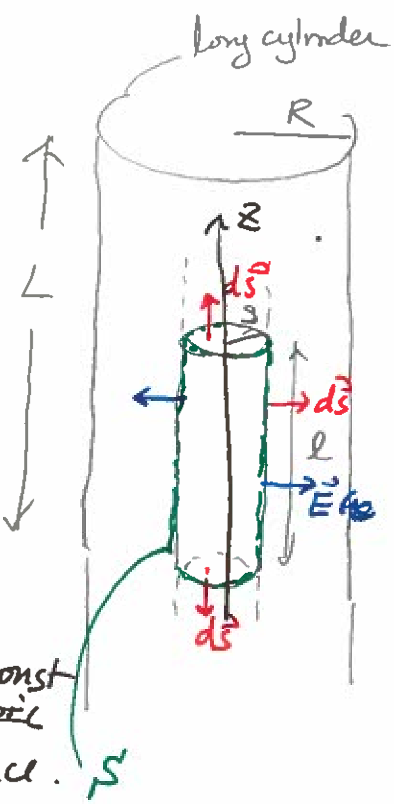
$$\therefore \vec{E}(\vec{r}) = |E(r)|\hat{r} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$$

EX $\vec{E}(r)$ for $r<R$ from a cylindrical charge with $\rho(s) = ks$.

Q_{enc} by the Gaussian surface S (green)

$$Q_{enc} = \int \rho\, d\tau = \int_0^s ks'\,ds'(2\pi s)\cdot \ell$$

$$= \frac{2}{3}\pi\ell k s^3$$



By symmetry, $\vec{E}$ is in $\hat{s}$ and ~~isotropic~~ $|\vec{E}|=\text{const}$ on the cylindrical surface.

$$\therefore \oint_S \vec{E}\cdot d\vec{s} = \int_{cylin} \vec{E}\cdot d\vec{s} + \underbrace{\int_{bottom} \vec{E}\cdot d\vec{s} + \int_{top} \vec{E}\cdot d\vec{s}}_{\vec{E}\perp d\vec{s}}$$

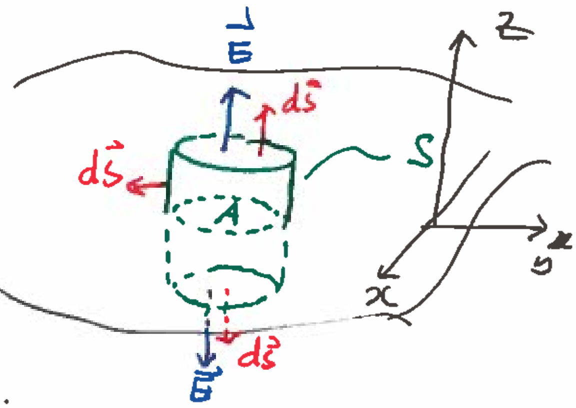
$$= \int_{\text{cylind}} |\vec{E}|\, ds = 2\pi s \ell |\vec{E}| = \frac{2}{3}\pi \ell k s^3/\varepsilon_0 .$$

$$\therefore \vec{E}(\vec{r}) = |\vec{E}|\hat{s} = \frac{1}{3\varepsilon_0} k s^2 \hat{s}$$

Ex $\vec{E}(\vec{r})$ from a uniformly charged infinite sheet with σ.

$Q_{enc} = \sigma \cdot A$

- $\vec{E}$ ∥ in $d\vec{s}$ for top and bottom.
- $\vec{E} \perp d\vec{s}$ on the cylindrical surf.



$$\therefore \oint_S \vec{E}\cdot d\vec{s} = |\vec{E}|\cdot A + |\vec{E}|\cdot A$$
$$= 2|\vec{E}|A = \sigma A/\varepsilon_0 .$$

$$\therefore \vec{E} = |\vec{E}|\, \text{sign}(z)\, \hat{z} = \underbrace{\frac{\sigma}{2\varepsilon_0}}_{}\text{sign}(z)\,\hat{z} \text{ or } \frac{\sigma}{2\varepsilon_0}\hat{n} .$$

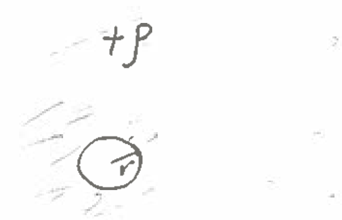
uniform field
no $\vec{r}$-dep.

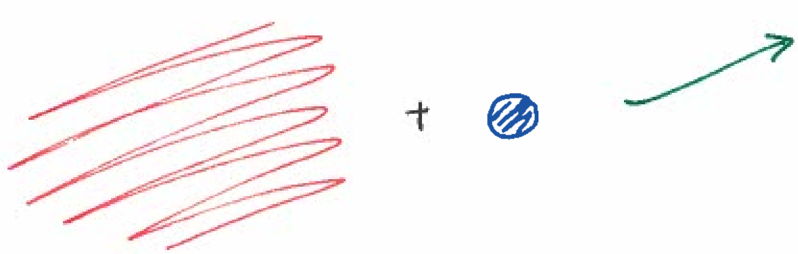
HW 2.11, 2.12, 2.14, 2.16, & $\boxed{2.18}$

* In the sea of uniform charge with $+\rho$ (positive), create a small spherical uncharged space.

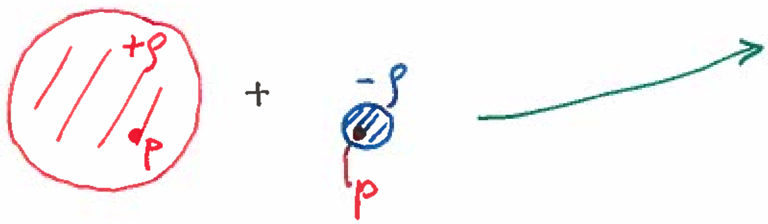
$\Rightarrow$ uniform charge with $+\rho$

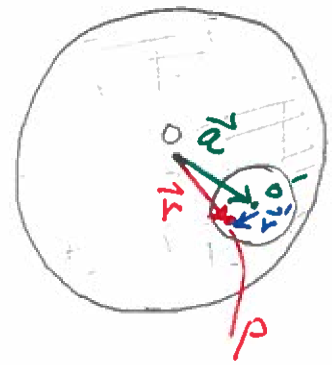
+ uniform charge sphere of r with $-\rho$





Ex Uniformly charged sphere of R with $+\rho$.
spherical cutout of S.
$\vec{E}$ field at point P inside the cutout.





$\vec{E}_+(\vec{r})$: field from (+) at $\vec{r}$
$\vec{E}_-(\vec{r})$: field from (−) at $\vec{r}$
$\vec{E}_t(\vec{r}) = \vec{E}_+(\vec{r}) + \vec{E}_-(\vec{r})$.

$$\vec{E}_+(\vec{r}) = \frac{\rho r}{3\varepsilon_0}\hat{r} \quad \text{and} \quad \vec{E}_-(\vec{r}) = \frac{-\rho r'}{3\varepsilon_0}\hat{r}'$$

where $\vec{r}' = \vec{r} - \vec{a}$ $\quad$ $\vec{a} = \vec{OO'}$

$$\therefore \vec{E}_t(\vec{r}) = \frac{\rho}{3\varepsilon_0}\left(\vec{r} - (\vec{r} - \vec{a})\right) = \frac{\rho}{3\varepsilon_0}\underline{\underline{\vec{a}}}$$

↳ given by the geometry.
inside the empty sphere $\vec{E}$ is const.!

$\vec{\nabla} \times \left(\frac{\hat{r}}{r^2}\right) = 0$ and $V(\vec{r})$

$\therefore$ $\boxed{\vec{\nabla} \times \vec{E}(\vec{r}) = 0}$ for $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \Rightarrow \vec{E} = \vec{\nabla} f(\vec{r})$.
$\vec{\nabla} \times \vec{\nabla} = 0$.

$$\therefore \oint_C \vec{E} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{s} = 0.$$

$\Downarrow$

$$\underset{(i)}{\int_a^b} \vec{E} \cdot d\vec{l} + \underset{(ii)}{\int_b^a} \vec{E} \cdot d\vec{l} = \underset{(i)}{\int_a^b} \vec{E} \cdot d\vec{l} - \underset{(ii)}{\int_a^b} \vec{E} \cdot d\vec{l} = 0$$

$$\therefore \underset{(i)}{\int_a^b} \vec{E} \cdot d\vec{l} = \underset{(ii)}{\int_a^b} \vec{E} \cdot d\vec{l}.$$

independent of path between two fixed pts.!

$\vec{E}$ is a conservative field!

Therefore we can define a scalar field $V(\vec{r})$.

$$V(\vec{r}) = -\int_{\underset{= \text{reference pt.}}{\mathcal{O}}}^{\vec{r}} \vec{E}(\vec{r})\cdot d\vec{l} \quad : \text{ Electric Potential}$$

<u>Notes</u>

(i) The absolute value of $V(\vec{r})$ depends on $\mathcal{O}$.

(ii) But the difference of V between two points is universal.

$$\left.\begin{aligned} V(\vec{b}) &= \int_{\mathcal{O}}^{\vec{b}} \vec{E}\cdot d\vec{l} \\ V(\vec{a}) &= \int_{\mathcal{O}}^{\vec{a}} \vec{E}\cdot d\vec{l} \end{aligned}\right\} \rightarrow V(\vec{b}) - V(\vec{a}) = -\int_{\mathcal{O}}^{\vec{b}} \vec{E}\cdot d\vec{l} + \int_{\mathcal{O}}^{\vec{a}} \vec{E}\cdot d\vec{l}$$

$$= -\int_{\mathcal{O}}^{\vec{b}} \vec{E}\cdot d\vec{l} - \int_{a}^{\mathcal{O}} \vec{E}\cdot d\vec{l}$$

$$\boxed{V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l}}$$

$$= \int_{\vec{a}}^{\vec{b}} \vec{\nabla} V(\vec{r})\cdot d\vec{l}$$

$$\boxed{\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})} \Leftrightarrow \boxed{\vec{\nabla}\times\vec{E} = 0.}$$

$\downarrow$

$$V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{E}(\vec{r}')\cdot d\vec{l}'$$

Reference point is set $V(\infty) = 0$ at $\vec{r} \to \infty$ when no charge is present at $\vec{r} \to \infty$ (localized charge dist.)

<u>Ex</u> $V(\vec{r})$ for $r > R$ and $r < R$ from a spherical shell of R with σ (const).

Using Gauss's law

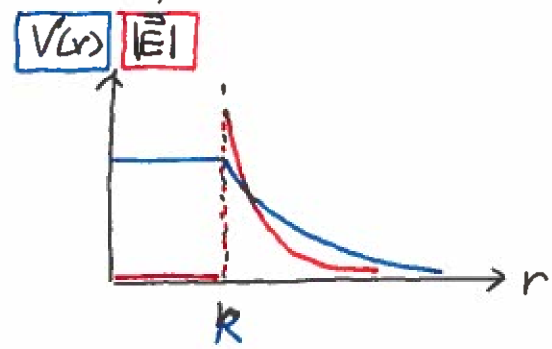
$$\vec{E}(\vec{r}) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2}\hat{r} & (r > R) \quad \text{where } q = 4\pi R^2 \sigma \\ 0 & (r < R) \end{cases}$$

$$\therefore \quad V(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{E}(\vec{r}')\cdot d\vec{l}' = -\int_{\infty}^{r} \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}\,dr = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}. \quad (r>R)$$

For $r<R$

$$V(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{E}(\vec{r}')\,d\vec{l} = -\int_{\infty}^{\vec{R}} \vec{E}(\vec{r}')\cdot d\vec{l} + \underbrace{\int_{\vec{R}}^{\vec{r}} \vec{0}\cdot d\vec{l}}_{0}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{R}.$$



↳ discontinuity in E at R (charge).

HW 2.20(b), 2.21, 2.22, & 2.24.
HW 2.50

In general, for a localized charge dist.

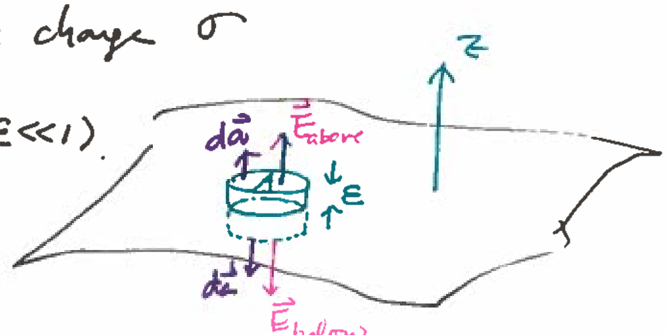
$$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}}\,d\tau'}$$

Easier than $\vec{E}(\vec{r})$ calculation.

HW 2.25 only for the disc. 2.26 & 2.27.

▸ Important Consequences from $\left\langle \begin{array}{l} \vec{\nabla}\cdot\vec{E} = \rho(\vec{r})/\varepsilon_0 \\ \vec{\nabla}\times\vec{E} = 0 \end{array} \right\rangle$.

∘ Consider a sheet of surface charge σ
Imagine a small pillbox of area A and height 2ε ($\varepsilon \ll 1$) as shown in the figure.



Gauss's thm

$$\int_V (\vec{\nabla}\cdot\vec{E})\,d\tau = \oint_S \vec{E}\cdot d\vec{a} = \sigma A/\varepsilon_0$$

$$(\text{as } \varepsilon\to 0). \quad = \vec{E}_{above}\cdot\hat{z}(A) + \vec{E}_{below}\cdot(-\hat{z})(A)$$

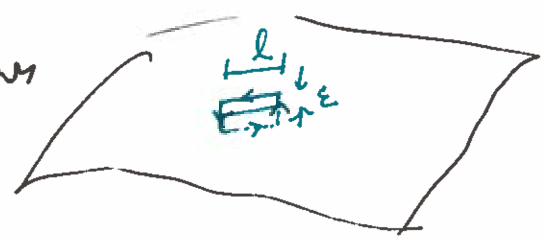
$$\vec{E}_{above} = E^{\perp}_{above}\,\hat{z} \quad (\neq 0)$$
$$\vec{E}_{below} = E^{\perp}_{below}\,\hat{z} \quad (\neq 0)$$

$$= E^{\perp}_{above} - E^{\perp}_{below}$$

$$\therefore \boxed{E^{\perp}_{above} - E^{\perp}_{above} = \frac{\sigma}{\varepsilon_0}}$$

For $\sigma = 0$, $E^{\perp}$ is continous.

- Imagine a loop of length l and width 2ε ($\varepsilon \ll 1$) as shown in the figure.



$$\int_S (\vec\nabla \times \vec E) \cdot d\vec a = \oint_C \vec E \cdot d\vec l$$

$$= (E^{\parallel}_{above} - E^{\parallel}_{below}) \cdot l \quad (\text{as } \varepsilon \to 0)$$

$$= 0 .$$

$\therefore$ $E^{\parallel}$ is always continous!

Therefore, combining the above,

$$\boxed{\vec E_{above} - \vec E_{below} = \frac{\sigma}{\varepsilon_0} \hat n}$$

surface normal ((+) above, (−) below).

Or since ~~$V(\vec r) = -$~~ $\vec E = -\vec\nabla V(\vec r)$,

$$\vec\nabla V_{above} - \vec\nabla V_{below} = -\frac{\sigma}{\varepsilon_0} \hat n .$$

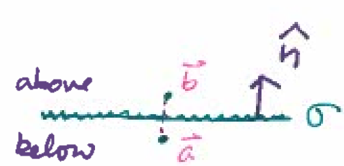
$$\hat n \cdot (\vec\nabla V_{above} - \vec\nabla V_{below}) = -\frac{\sigma}{\varepsilon_0} .$$

$$\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0} \qquad \left(\hat n \cdot \vec\nabla V = \frac{\partial V}{\partial n} \right)$$

"normal derivative"

However,

$$V_{above} - V_{below} = -\int_{\vec a}^{\vec b} \vec E \cdot d\vec l$$



$$= 0 \quad \text{as } \vec b - \vec a \to 0 .$$

$$\therefore V_{above} = V_{below} .$$