$\boxed{11.6}$

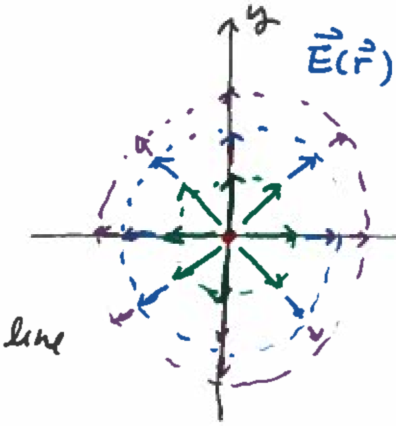
- Gauss's Law
  - E-field lines and $\vec{E}$

vector field.

$\vec{E}$ from a point charge q at $\vec{r}=0$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2}\hat{r}$$

$\Downarrow$



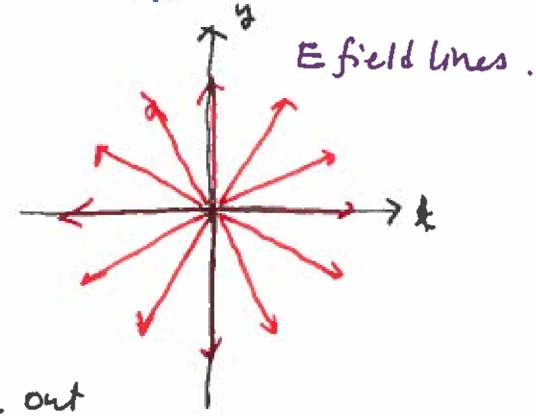
Field lines : a set of continuous lines w/ direction.

(i) $\hat{E}$-direction : tangential direction of the field line

(ii) $|\vec{E}|$: density of the field lines near the point $\vec{r}$

(iii) Nature made the representation work extremely well!

Coulomb's Law



$\vec{E}$ from a point charge is spherically symmetric!
Therefore, E-field lines should come out of the pt. charge radially & uniformly. But the direction depends on the type of the charge. Then the $|\vec{E}|$ comes naturally from the geometry.

Set the total # of field lines as N. The line density through spherical surfaces of radius r_1 & r_2 can be written

$$n_1 = \frac{N}{4\pi r_1^2} \quad \text{and} \quad n_2 = \frac{N}{4\pi r_2^2}$$

$$\frac{|E(\vec{r}_2)|}{|E(\vec{r}_1)|} = \frac{r_1^2}{r_2^2} = \frac{n_2}{n_1}$$

Notes

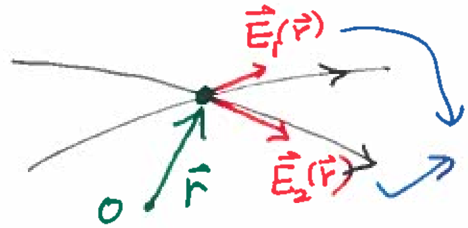
(i) The total # of field lines per unit charge must be set.

(ii) Field lines come out of (+)-charge : source of field lines

enter into (−)-charge : sink of field lines.

(iii) E-field lines can not end or start in empty space pt.

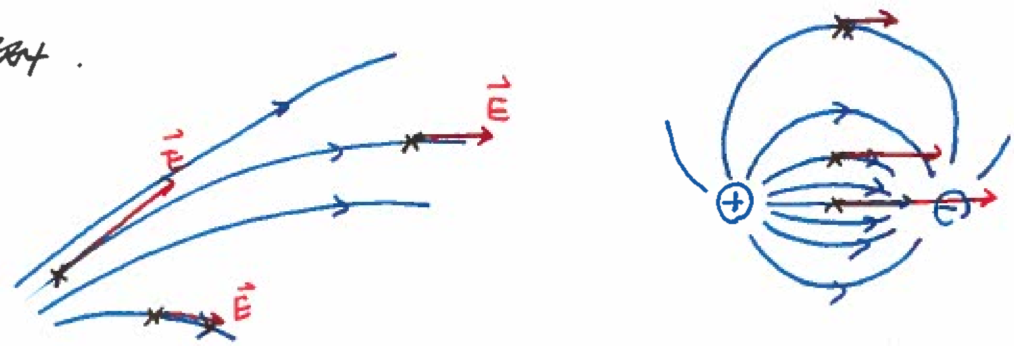
(iv) E-field lines can not cross each other.



two different $\vec{E}$s at $\vec{r}$ (X)

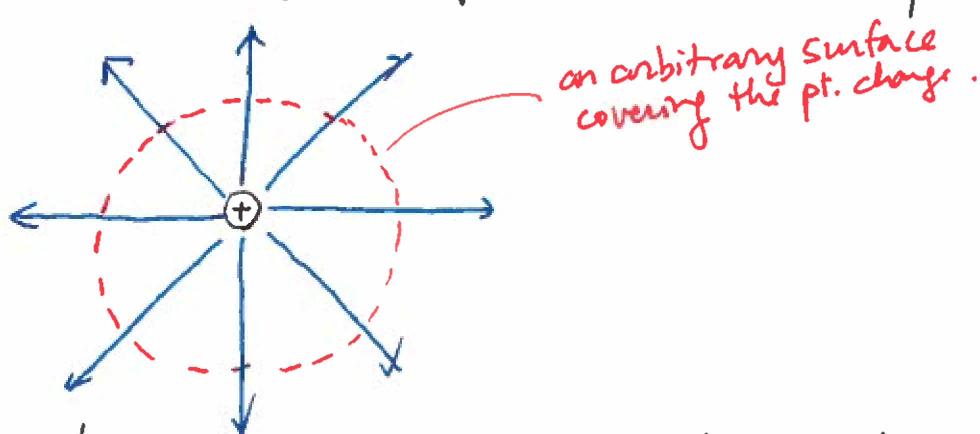
~~Ex~~

Ex

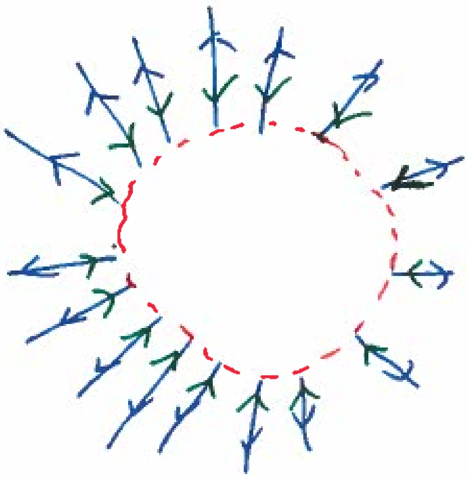


⊙ Flux.

Let's set 8 total field lines come out of a unit (+) charge.



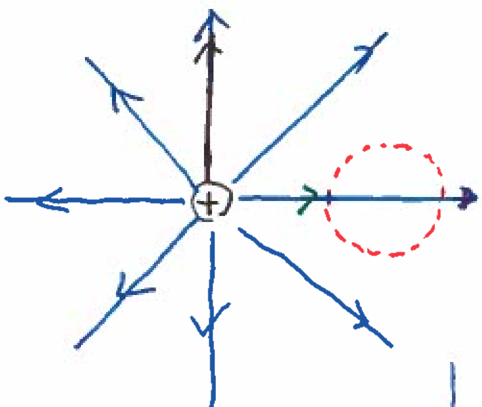
There are 8 lines penetrating through the surface.



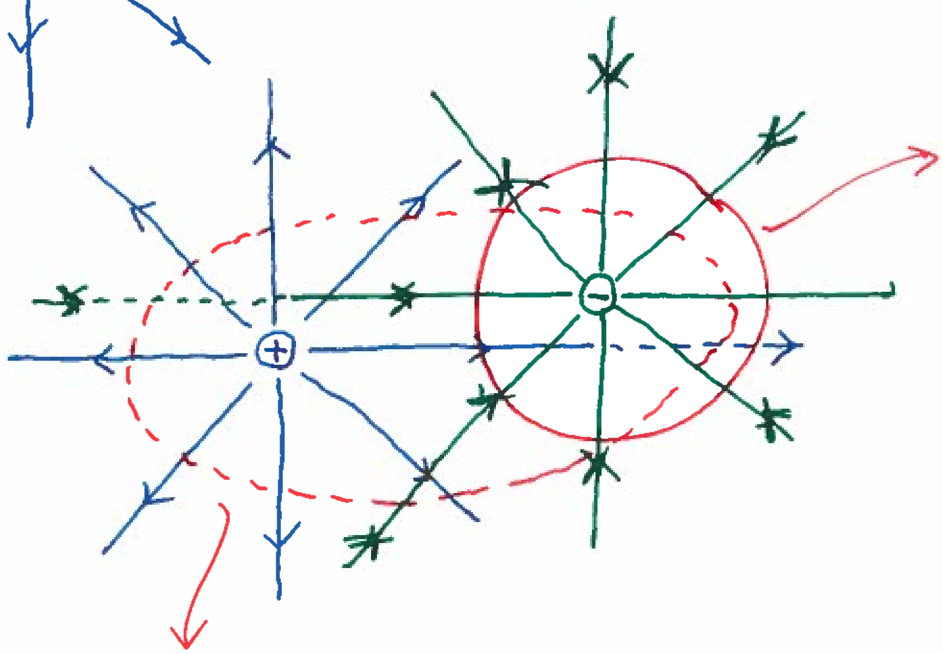
16 lines are penetrating through the surface.
What charge is hidden inside?

$\frac{16}{8} = 2 \Rightarrow 2\oplus$ for $\nearrow$

$2\ominus$ for $\swarrow$



$\nearrow + \swarrow = 0 \Rightarrow$ no net charge inside



$8 \swarrow = \ominus$

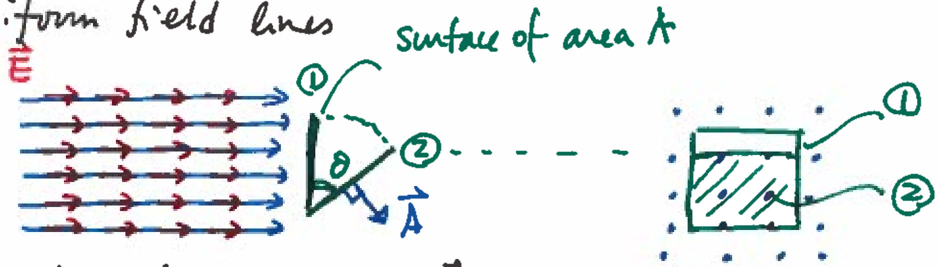
$8 \nearrow + 8 \swarrow = 0$ (net) charge.

$$\boxed{\text{\# of net field lines through a closed surface} \Longleftrightarrow \text{Amount of net charge enclosed by the surface}}$$

Gauss's Law.

◎ Flux : Amount of vector field penetrating through a surface.

Uniform field lines



Flux through ① $\quad \Phi_1 = |\vec{E}| \cdot A$

Flux through ② $\quad \Phi_2 = |\vec{E}| \underbrace{A\cos\theta}_{\text{effective area}}$

$\Phi = \vec{E} \cdot \underline{\underline{\vec{A}}}$

surface vector

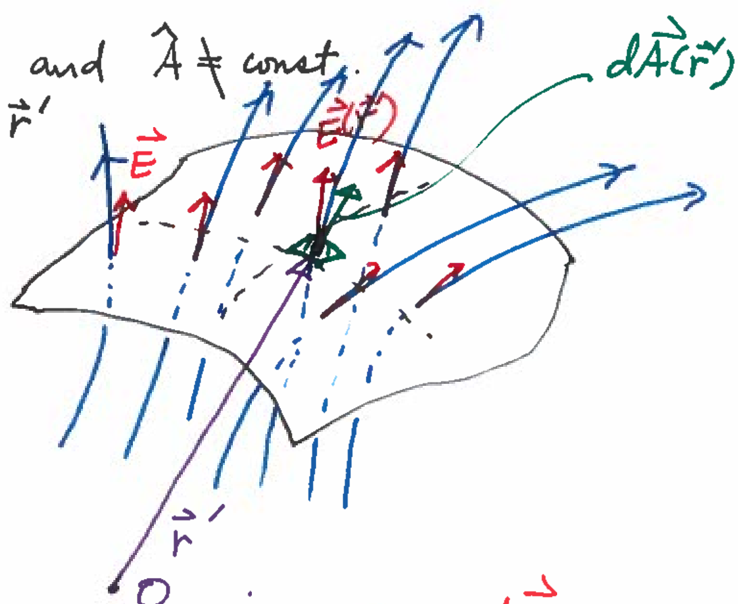
$|\vec{A}| = \text{area}$ $\quad \hat{A}$: surface normal vector.

In general, $\vec{E} = \vec{E}(\vec{r})$ and $\hat{A} \neq \text{const}$.

Then the local flux at $\vec{r}\,'$ should be defined as

$$d\Phi_E = \vec{E}(\vec{r}\,') \cdot d\vec{A}(\vec{r}\,')$$

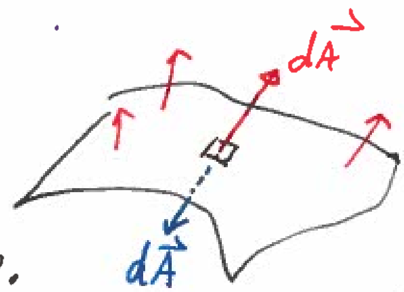
$$\Phi_E = \int_S \vec{E}(\vec{r}\,') \cdot d\vec{A}(\vec{r}\,')$$



<u>Notes</u>

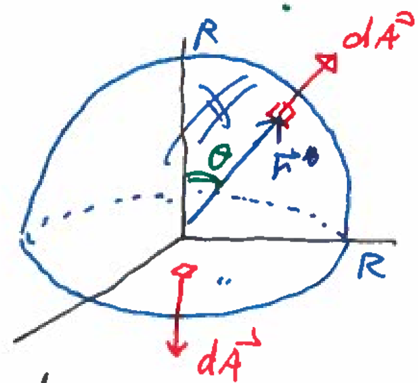
Choice of $d\vec{A}$

(i) For an open surface, either choice works. Just be consistent to choose one side.



(ii) For a closed surface, the convention is outward direction

Ex Calculate Φ for the closed surface of a hemisphere of radius R. $(z \geq 0)$
$\vec{E}(\vec{r}) = E_0 \hat{z}$ (uniform field).



(i) bottom flat surface

$d\vec{A} = dx\,dy\,(-\hat{z})$: no position depend.

$$\therefore \Phi_B = \vec{E}\cdot\vec{A} = E_0\hat{z}\cdot(-\pi R^2)\hat{z} = -\pi R^2 E_0$$

(ii) top spherical surface

$$\vec{E}\cdot d\vec{A}(\vec{r}) = \vec{E}\cdot dS\,\hat{r} = E_0 R^2 \sin\theta\, d\varphi\, d\theta \underbrace{\hat{z}\cdot\hat{r}}_{\cos\theta}.$$

$$\therefore \Phi_S = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} E_0 R^2 \sin\theta\cos\theta\, d\theta\, d\varphi$$

$$= 2\pi E_0 R^2 \int_0^{\pi} \sin\theta\cos\theta\, d\theta \qquad \left(\begin{aligned}\sin\theta &\equiv t\\ d\theta\cos\theta &= dt\end{aligned}\right)$$

$$= 2\pi E_0 R^2 \int_0^1 t\,dt = +\pi R^2 E_0$$

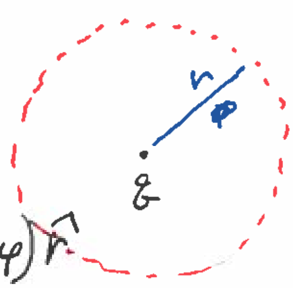
$$\therefore \Phi_{tot} = \Phi_B + \Phi_S = -\pi R^2 E_0 + \pi R^2 E_0 = \underline{\underline{0}}$$

◦ **Gauss's Law**

Consider a point charge q at $\vec{r} = 0$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}.$$

Flux of $\vec{E}$ through a spherical surface of radius r : Φ_r



$$\Phi_r = \oint_S \vec{E}\cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0}\oint_S \frac{1}{r^2}\hat{r}\cdot(r^2\sin\theta\, d\theta\, d\varphi)\hat{r}$$

$$= q/\varepsilon_0$$

$\Phi_r \propto q$

ε_0 : scaling factor for dimensional correctness.

physical dimension.

$$[\Phi] = [EA] = \left[\frac{q}{4\pi\varepsilon_0 r^2}\right][A] = \left[\frac{q}{\varepsilon_0}\right]\left[\frac{A}{r^2}\right] = \left[\frac{q}{\varepsilon_0}\right].$$

$$\Phi = q/\varepsilon_0 = \oint_S \vec{E}\cdot d\vec{s}$$

→ for any closed surface enclosing charge q

the charge cannot be "on" the surface!

For multiple charges $(q_1, \cdots, q_n)$ at $(\vec{r}_1', \cdots \vec{r}_n')$.

$$\vec{E}(\vec{r}) = \sum_{i=1} \vec{E}_i(\vec{r}) = \sum_{i=1} \frac{1}{4\pi\varepsilon_0}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i \quad \text{where } \vec{\mathscr{r}}_i = \vec{r} - \vec{r}_i'$$

Flux through an arbitrary surface that encloses all

$$\boxed{\Phi = \oint_S \vec{E}\cdot d\vec{s} = \sum_{i=1}^{n}\oint_S \vec{E}_i\cdot d\vec{s} = \sum_{i=1}^{n}\left(\frac{q_i}{\varepsilon_0}\right) = \frac{Q_{enc}}{\varepsilon_0}.}$$

In general, $Q_{enc} = \int_V \underbrace{\rho}_{=\text{charge density}} d\tau$

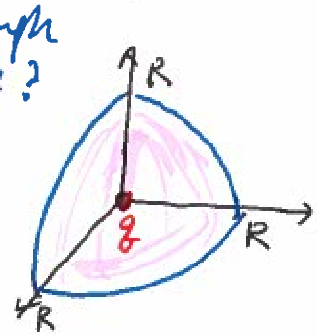
$$\therefore \oint_S \vec{E}\cdot d\vec{s} = \int_V \vec{\nabla}\cdot\vec{E}\, d\tau = \int_V \rho\, d\tau/\varepsilon_0.$$

$$\therefore \boxed{\vec{\nabla}\cdot\vec{E}(\vec{r}) = \rho(\vec{r})/\varepsilon_0.}$$

HW Prob. 2.9, 2.10, 2.46

→ Do you really need to do the full calculation?

* what is the flux through the surface of 1/8 sphere?



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\, d\tau'$$

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}')\, \vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau'$$

$$\vec{\nabla}_r\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \vec{\nabla}_{\mathscr{r}}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\vec{\mathscr{r}})$$

$$\therefore\ \vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}')\, 4\pi\delta^3(\vec{\mathscr{r}})\, d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}')\, 4\pi\delta(\hat{r}-\vec{r}')\, d\tau'$$

$$= \rho(\vec{r})/\varepsilon_0 .$$