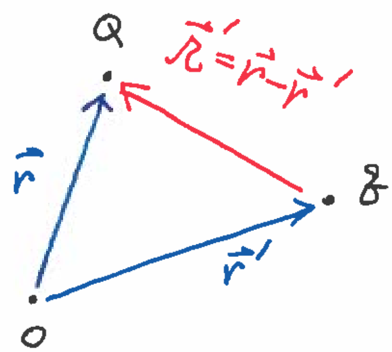
Ch. 2 Electrostatics

- Coulomb's Law (Coulomb force)

$$\vec{F}_Q = \frac{1}{4\pi\varepsilon_0}\cdot\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}} \quad \text{where } \vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$$



Action-reaction pair

$$\vec{F}_q = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\mathscr{r}^2}(-\hat{\mathscr{r}}) = -\vec{F}_Q .$$

ε_0: permittivity of vacuum (free space)

$$= 8.85\times10^{-12} \ C^2/N\cdot m^2 .$$

$$\frac{1}{4\pi\varepsilon_0} = 9\times10^{9} \ \frac{Nm^2}{C^2}$$

* Force on a charge Q by multiple point charges $(q_1, \cdots, q_n)$

$$\vec{F}_Q = \sum_{i=1}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_i Q}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i \quad \text{where } \vec{\mathscr{r}}_i = \vec{r} - \vec{r}_i \ ; \ \vec{r}_i = \text{position of } q_i$$

- Electric Field $\vec{E}$

$$\vec{F}_Q = \boxed{Q}\vec{E} \Rightarrow \vec{E} = \sum_{i=1}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i$$

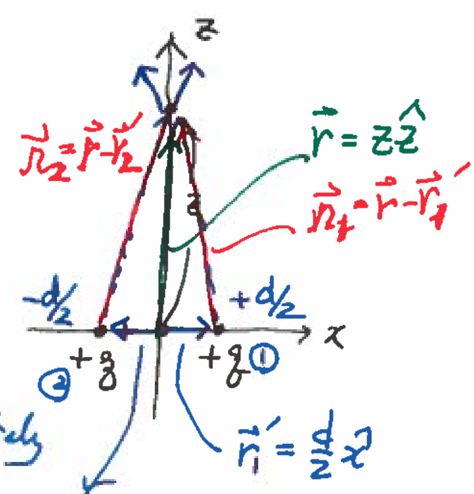
E is determined by other charges and does not depend on a specific charge Q.

EX

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + \frac{q}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{z\hat{z}+\frac{d}{2}\hat{x}}{[z^2+(\frac{d}{2})^2]^{3/2}} + \frac{z\hat{z}-\frac{d}{2}\hat{x}}{[z^2+(\frac{d}{2})^2]^{3/2}}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2qz}{[z^2+(\frac{d}{2})^2]^{3/2}}\hat{z}$$



But by symmetry you know immediately the x-comp will be cancelled! ~~and~~
Calculate E_z from one charge and double it!

For $z \gg \frac{d}{2}$, (small parameter $\alpha = \frac{d}{2z} \ll 1$).

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{2q\cancel{z}}{z^{\cancel{3}2}}\cdot\frac{1}{\{1+(\frac{d}{2z})^2\}^{3/2}} = \frac{1}{4\pi\varepsilon_0}\frac{2q}{z^2}\frac{1}{(1+\alpha^2)^{3/2}}$$

$$\frac{1}{(1+\varepsilon)^{3/2}} \approx 1 - \frac{3}{2}\varepsilon + O(\varepsilon^2)$$

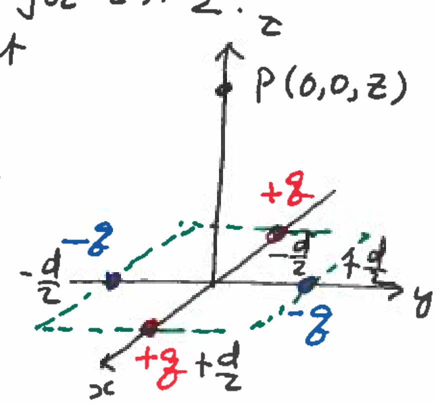
$$\therefore \vec{E}(\vec{r}) \approx \frac{1}{4\pi\varepsilon_0}\underbrace{\frac{2q}{z^2}}_{\text{just } 2q \text{ at the origin}}\left(1 - \underline{\frac{3}{2}\left(\frac{d}{2z}\right)^2}\right).$$

correction for finite but small $\frac{d}{2z}$

$z \not\to \infty$, $d \not\to 0$.

HW Prob. 2.2. Also consider the case for $z \gg \frac{d}{2}$.

HW Calculate $\vec{E}$ at P from 4 point charges shown in the figure. Consider the case for $z \gg \frac{d}{2}$.



- Continuous charge Distribution.
  - Point charge q in 3D space located at $\vec{r}=0$.

$q\delta^3(\vec{r}) = \rho(\vec{r})$: charge density distribution.

$$\int_V \rho(\vec{r})\,d\tau = \int_V q\delta^3(\vec{r})\,d\tau = q.$$

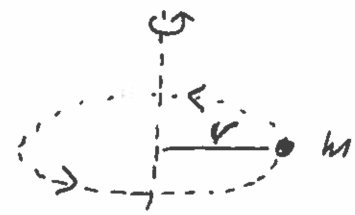
- Two point charges : $+q$ at $\vec{r}=\vec{a}$ and $-q$ at $\vec{r}=-\vec{a}$

$$\rho(\vec{r}) = q\delta^3(\vec{r}-a) + (-q)\delta^3(\vec{r}+\vec{a}).$$

$$\therefore \int_V \rho(\vec{r})\,d\tau = +q - q = 0.$$

○ Moment of Inertia

– a point mass m rotating around $\hat{z}$-axis at a distance r from the axis.



$$I = r^2$$

– a uniform ring of mass m and radius r rotating around its symmetric axis

Infinitesimal mass element dm = point mass

$$\Rightarrow dI = r^2 dm.$$

$$\therefore I = \int dI = \int \underbrace{r^2}_{\text{const.}} dm \Rightarrow \lambda \cdot ds$$

λ: mass line density $\lambda = \frac{m}{2\pi r}$

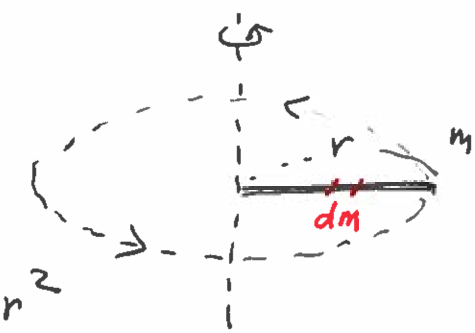
ds: infinitesimal arc length

$\therefore ds = r d\varphi$.

$$= r^2 \int_0^{2\pi} \frac{m}{2\pi \cancel{r}} \cdot \cancel{r} d\varphi$$

$$= m r^2$$

– a thin rod of length r and mass m



$$dm = \lambda dl = \left(\frac{m}{r}\right) dl.$$

$$\therefore dI = l^2 dm = l^2 \left(\frac{m}{r}\right) dl$$

$$I = \int dI = \int_0^r l^2 \underbrace{\left(\frac{m}{r}\right)}_{\text{const}} dl = \frac{1}{3} m r^2$$

○ $\vec{E}$ from continuous charge distribution.

– $d\vec{E}$ from infinitesimal charge element dq then $\vec{E} = \int d\vec{E}$ where

$$d\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{dq}{r^2} \hat{r}$$

– dq needs to expressed in terms of proper $\underbrace{\text{coordinates}}_{\text{integral variable}}$

For a line charge (1D)

$dq = \lambda(\vec{x})\,dx$ where $\lambda(\vec{x})$ is the line charge density. $[\lambda] = C/m$

$\lambda(\vec{x}) = \text{const}$ for a uniform line charge.

total charge $Q = \int dq = \int \lambda(x)\,dx$.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\vec{r}')}{\mathscr{r}^2}\,\hat{\mathscr{r}}\,dl' \qquad \lambda(\vec{r}')\,dl' = dq$$

For a surface charge (2D)

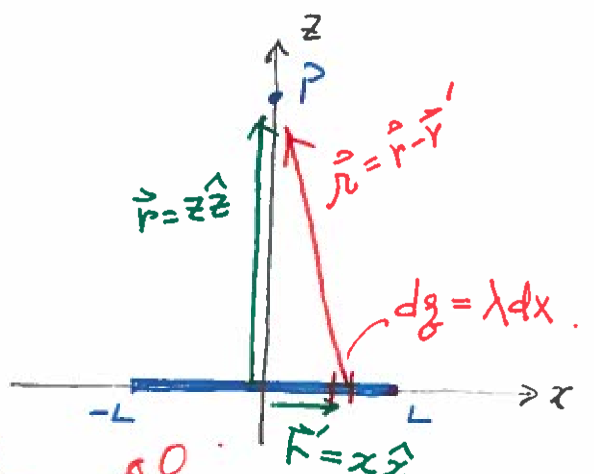
$dq = \sigma(\vec{r})\,ds$ where $\sigma(\vec{r})$ is the surface charge density. $[\sigma] = C/m^2$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\vec{r}')}{\mathscr{r}^2}\,\hat{\mathscr{r}}\,ds'$$

For a volume charge (3D)

$dq = \rho(\vec{r})\,d\tau$ where $\rho(\vec{r})$ is the vol. charge density $[\rho] = C/m^3$.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}^2}\,\hat{\mathscr{r}}\,d\tau'$$



Ex

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\vec{r}')\,dl'}{\mathscr{r}^2}\,\hat{\mathscr{r}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\int_{-L}^{L} \frac{dx\,(z\hat{z} - x\hat{x})}{(z^2+x^2)^{3/2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[\underbrace{\int_{-L}^{+L} \frac{z\,dx}{(z^2+x^2)^{3/2}}}_{\text{even fct}}\,\hat{z} - \cancel{\underbrace{\int_{-L}^{+L} \frac{x\,dx}{(z^2+x^2)^{3/2}}}_{\text{odd fct}}\,\hat{x}}\right] \quad \to 0$$

by symmetry you know that no x-comp.

$$\int_{-L}^{+L} \frac{z\,dx}{(z^2+x^2)^{3/2}} = 2z\int_0^L \frac{dx}{(x^2+z^2)^{3/2}} = 2z \cdot \left.\frac{x}{z^2\sqrt{x^2+z^2}}\right|_0^L$$

$$= \frac{2}{z}\cdot\frac{L}{\sqrt{z^2+L^2}}$$

total charge Q

$$\therefore\ \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{2\lambda L}{z}\right)\cdot\frac{1}{\sqrt{z^2+L^2}}\,\hat{z}$$

$$\int \frac{x\,dx}{(x^2\pm a^2)^n} = \frac{-1}{2(n-1)(x^2\pm a^2)^{n-1}}$$

$$\int \frac{dx}{(x^2\pm a^2)^{3/2}} = \frac{\pm x}{a^2\sqrt{x^2\pm a^2}}$$

$$\frac{1}{(1+\varepsilon^2)^{1/2}} \approx 1-\frac{1}{2}\varepsilon^2$$

For $z \gg L$,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{2\lambda L}{z^2}\right)\frac{1}{\left(1+\left(\frac{L}{z}\right)^2\right)^{1/2}}\,\hat{z}$$

$$\approx \frac{1}{4\pi\epsilon_0}\left(\frac{2\lambda L}{z^2}\right)\left(1-\frac{1}{2}\left(\frac{L}{z}\right)^2\right)\hat{z}$$

$$\approx \frac{1}{4\pi\epsilon_0}\frac{Q}{z^2} + \text{correction}. \quad \propto \frac{1}{z^2}$$

For $L \to \infty$,

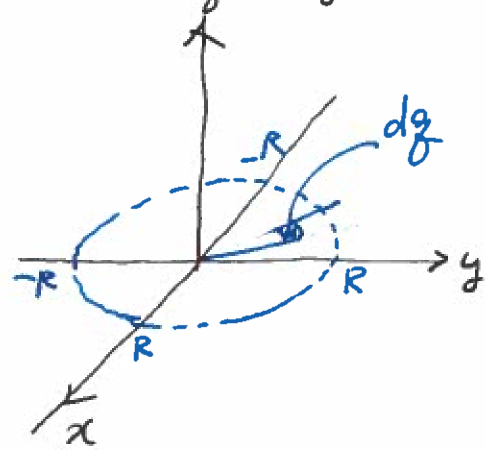
$$\vec{E}(\vec{r}) = \lim_{L\to\infty}\frac{1}{4\pi\epsilon_0}\left(\frac{2\lambda \cancel{L}}{z}\right)\left(\frac{1}{\cancel{L}}\right)\cdot\frac{1}{\sqrt{1+\left(\frac{z}{L}\right)^2}} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{z} \propto \frac{1}{z}.$$

HW Prob. 2-3, 2-4, 2-5, 2-6, 2-7, & 2-8.

- For 2.6, there are two ways of calculating $\vec{E}$ from a circular surface charge.
  (i) Using the result of 2.5, $d\vec{E}$ from a ring charge
  (ii) $d\vec{E}$ from a point charge.

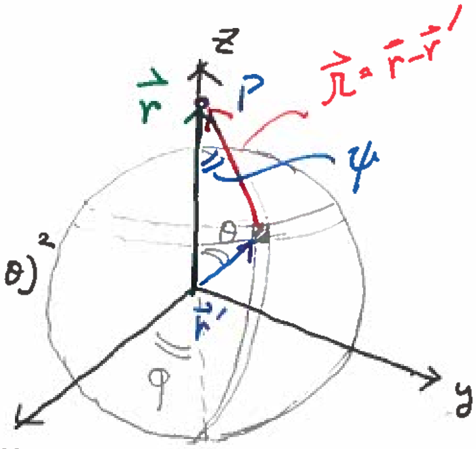
$$dq = \sigma\, ds$$
$$= \sigma \cdot r\,d\varphi\,dr$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi \int_0^R dr'\, \frac{\sigma r'}{\mathscr{r}^2}\hat{\mathscr{r}}$$



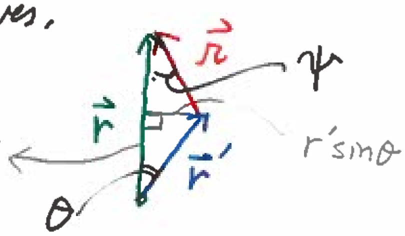
• Prob 2.7

See the figures.



$$\mathscr{r}^2 = (r - r'\cos\theta)^2 + (r'\sin\theta)^2$$
$$= r^2 - 2rr'\cos\theta + (r'\cos\theta)^2 + (r'\sin\theta)^2$$
$$= z^2 - 2zr'\cos\theta + r'^2$$
$$= z^2 - 2zR\cos\theta + R^2.$$

By symmetry, only $\hat{z}$-comp survives.



$$E_z = E\cos\psi = \frac{z - R\cos\theta}{\mathscr{r}}$$

$$dq = \sigma R^2 \sin\theta \, d\theta \, d\varphi.$$