$\boxed{L4}$ Ch. 2.   L1

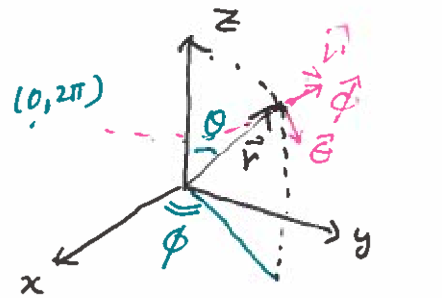
▶ Curvilinear Coordinates

⊙ Spherical Coordinates. (r, θ, φ) — polar angle $(0, \pi)$

– $x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$

azimuthal angle $(0, 2\pi)$

$$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\varphi\hat{\varphi}$$
$$= A_r\hat{e}_r + A_\theta\hat{e}_\theta + A_\varphi\hat{e}_\varphi$$



$$\hat{r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$
$$\hat{\theta} = \cos\theta\cos\varphi\,\hat{x} + \cos\theta\sin\varphi\,\hat{y} - \sin\theta\,\hat{z}$$
$$\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$$

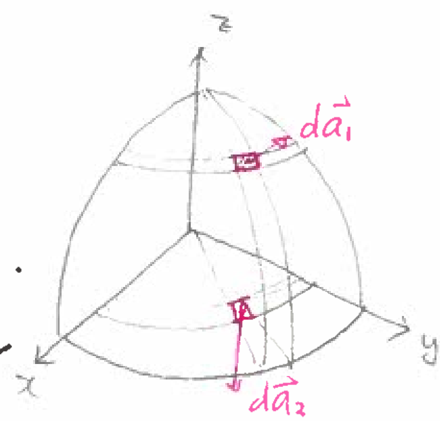
$$\hat{r}\cdot\hat{\theta} = \hat{\theta}\cdot\hat{\varphi} = \hat{r}\cdot\hat{\varphi} = 0$$
$$\hat{r}\times\hat{\theta} = \hat{\varphi}\ ;\quad \hat{\theta}\times\hat{\varphi} = \hat{r}\ ;\quad \hat{\varphi}\times\hat{r} = \hat{\theta}$$

"orthogonal coordinate system"

* all function of θ, φ. not fixed in space !

$$d\vec{r} = d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\varphi}$$
$$dv = d\tau = r^2\sin\theta\,dr\,d\theta\,d\varphi$$

$$d\vec{a}_1 = r^2\sin\theta\,d\theta\,d\varphi\,\hat{r}$$
$$d\vec{a}_2 = r\,dr\,d\varphi\,\hat{\theta}$$



EX Volume of a sphere by integration.

$$V = \int d\tau = \int_0^R\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} r^2\sin\theta\,d\theta\,d\varphi\,dr$$
$$= \int_0^R r^2 dr \int_0^{\pi}\sin\theta\,d\theta \int_0^{\pi} d\varphi$$
$$= \frac{1}{3}R^3\cdot 2\cdot\pi \quad = \frac{4}{3}\pi R^3$$

$$\vec{\nabla} F = \left(\frac{\partial F}{\partial x}\right)\hat{x} + \left(\frac{\partial F}{\partial y}\right)\hat{y} + \left(\frac{\partial F}{\partial z}\right)\hat{z}$$

$$= \square\, \hat{r} + \triangle\, \hat{\theta} + \bigcirc\, \hat{\varphi}$$

$$x = x(r,\theta,\varphi)\ ;\ y = y(r,\theta,\varphi)\ ;\ z = z(r,\theta).$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial x}.$$

$$\frac{\partial}{\partial y} = \cdots$$

$$\frac{\partial}{\partial z} = \cdots$$

$$\hat{x} = \odot\, \hat{r} + \boxdot\, \hat{\theta} + \triangle\, \hat{\varphi}$$

$$\hat{y} = \cdots$$

$$\hat{z} = \cdots$$

$$\boxed{\begin{array}{lll} \frac{\partial \hat{r}}{\partial r} = 0 & \frac{\partial \hat{\theta}}{\partial r} = 0 & \frac{\partial \hat{\varphi}}{\partial r} = 0 \\ \frac{\partial \hat{r}}{\partial \theta} = \hat{\theta} & \frac{\partial \hat{\theta}}{\partial \theta} = -r\hat{r} & \frac{\partial \hat{\varphi}}{\partial \theta} = 0 \\ \frac{\partial \hat{r}}{\partial \varphi} = \sin\theta\, \hat{\varphi} & \frac{\partial \hat{\theta}}{\partial \varphi} = \cos\theta\, \hat{\varphi} & \\ & \frac{\partial \hat{\varphi}}{\partial \varphi} = -\sin\theta\, \hat{r} - \cos\theta\, \hat{\theta} & \end{array}}$$

Finally,

$$\vec{\nabla} F = \frac{\partial F}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial F}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial F}{\partial \varphi}\hat{\varphi}$$

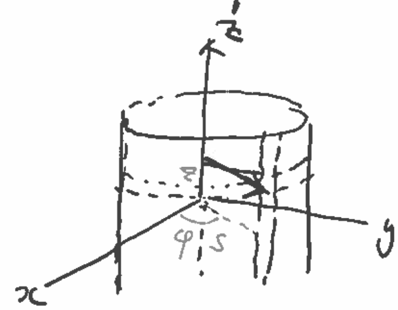
$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}\left(\sin\theta A_\varphi\right) - \frac{\partial A_\theta}{\partial \varphi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\varphi}$$

$$\nabla^2 F = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial F}{\partial \theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 F}{\partial \varphi^2}$$

⑥ Cylindrical Coordinates (s, φ, z)



$$x = s\cos\varphi\ ;\ y = s\sin\varphi\ ;\ z = z.$$

$$\vec{A} = A_s\hat{s} + A_\varphi\hat{\varphi} + A_z\hat{z} \qquad (\hat{s}\perp\hat{\varphi}\perp\hat{z})$$

$$\hat{s} = \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}$$

$$\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$$

$$\hat{z} = \hat{z}.$$

$$d\vec{r} = d\vec{\ell} = ds\,\hat{s} + s\,d\varphi\,\hat{\varphi} + dz\,\hat{z}$$
$$dV = d\tau = s\,ds\,d\varphi\,dz$$
$$\vec{\nabla}F = \left(\frac{\partial F}{\partial s}\right)\hat{s} + \frac{1}{s}\left(\frac{\partial F}{\partial \varphi}\right)\hat{\varphi} + \left(\frac{\partial F}{\partial z}\right)\hat{z}$$
$$\vec{\nabla}\cdot\vec{A} = \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$
$$\vec{\nabla}\times\vec{A} = \left(\frac{1}{s}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{s} + \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right)\hat{\varphi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sA_\varphi) - \frac{\partial A_s}{\partial \varphi}\right]\hat{z}$$
$$\nabla^2 F = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial F}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 F}{\partial \varphi^2} + \frac{\partial^2 F}{\partial z^2}.$$

○ Generalization of Coordinate Systems ("orthogonal")

- Suppose that every point in 3D space can be represented by three coordinates (u_1, u_2, u_3).
  u_i's are smooth functions of $\vec{r} = (x, y, z)$.

- The essence of any coordinate system is in the differential length interval, ds.

$$ds^2 = (h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^2$$

where h_i are non-negative scalar factor, varying smoothly with $\vec{r}$ or u_i

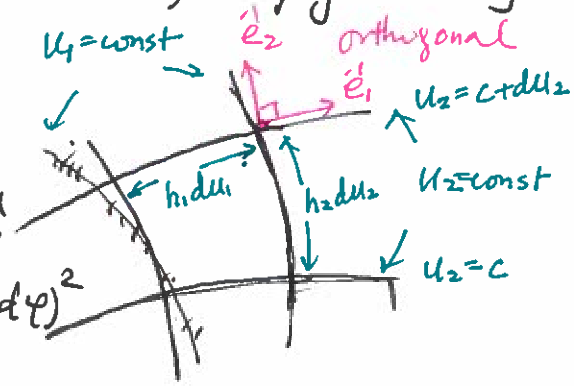
For example, in spherical coord.

~~$d\vec{\ell} = (dr\hat{r} + r\sin\theta\, d\theta$~~

$$d\vec{\ell} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\varphi\,\hat{\varphi}$$

$$\therefore (ds)^2 = d\vec{\ell}\cdot d\vec{\ell} = (dr)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\varphi)^2$$

$$\therefore h_r = 1 \; ; \; h_\theta = r \; ; \; h_\varphi = r\sin\theta$$



example of 2D curvilinear system

Also in ~~spheri~~ cylindrical coord.

$h_s = 1$; $h_\varphi = s$; $h_z = 1$.

- All vector derivatives can be written the following general forms.

$$\vec{\nabla} F = \frac{1}{h_1}\frac{\partial F}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial F}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial F}{\partial u_3}\hat{e}_3$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \cdots \right]$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial u_2}(h_3 A_3) - \frac{\partial}{\partial u_3}(h_2 A_2)\right]\hat{e}_1 + \cdots$$

$$\nabla^2 F = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial F}{\partial u_1}\right) + \cdots \right]$$

$\boxed{\text{HW}}$ Prob. 1.38 & 1.39

$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$

$= r\sin\theta\cos\varphi\,\hat{x} + r\sin\theta\sin\varphi\,\hat{y} + r\cos\theta\,\hat{z}$

$\underbrace{\hat{r} = \frac{d\vec{r}}{dr}}_{\text{meaning of the expression.}} \Big/ \left|\frac{d\vec{r}}{dr}\right|$; $\hat{\theta} = \frac{d\vec{r}}{d\theta}\Big/\left|\frac{d\vec{r}}{d\theta}\right|$; $\hat{\varphi} = \frac{d\vec{r}}{d\varphi}\Big/\left|\frac{d\vec{r}}{d\varphi}\right|$

$\boxed{\text{HW}}$ Prob. 1.43

$\boxed{\text{HW}}$ Prob. 1.62

▶ Dirac Delta function

⊙ 1D Dirac delta fct.

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

with $\displaystyle\int_{-\infty}^{+\infty} \delta(x)\,dx = 1$

- $\delta(-x) = \delta(x)$: symmetric fct.

- If $f(x)$ is continuous,

$$f(x)\delta(x) = f(0)\delta(x).$$

$$\therefore \int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0)\int_{-\infty}^{+\infty}\delta(x)dx = f(0).$$

And also

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)dx = f(a)\int_{-\infty}^{+\infty}\delta(x-a)dx = f(a).$$

- $\delta(kx) = \frac{1}{|k|}\delta(x)$

- $x\frac{d}{dx}(\delta(x)) = -\delta(x)$.

HW Prob. 1.44

○ 3D Dirac delta function.

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z) = \begin{cases} 0 & \text{otherwise} \\ \infty & \text{if } x=y=z=0 \end{cases}.$$

with $\int_{-\infty}^{+\infty} dx\,dy\,dz\,\delta^3(\vec{r}) = 1$.

In general for d-dim $\delta^d(\vec{r}) = \prod_{i=1}^{d}\delta^i(x_i)$

- $\int_V f(\vec{r})\delta^3(\vec{r}-\vec{a})dV = f(\vec{a})$

- $\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$ and $\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\vec{\mathscr{r}})$

We know $\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$

**

$$\therefore \boxed{\vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) = \nabla^2\left(\frac{1}{\mathscr{r}}\right) = -\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = -4\pi\delta^3(\vec{\mathscr{r}})}$$

- $\delta^3(k\vec{r}) = \frac{1}{|k|^3}\delta(\vec{r})$

* Early HW problem (1.16). $\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = \vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)$

In Cartesian Coordinate

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) + \cdots$$

But in sperical Coordinate

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{1}{r^2}\right) = 0 \quad \text{for } r \neq 0.$$

$\frac{\hat{r}}{r^2}$ is not well-defined at $r=0$.

$$\int_V \underbrace{\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)}\, d\tau = \oint_S \frac{\hat{r}}{r^2}\cdot d\vec{s}$$

($d\vec{s}$ on the spherical surface of radius $r \Rightarrow R^2 \sin\theta\, d\theta\, d\varphi\, \hat{r}$)

$$= \underbrace{\int_0^{\pi} \sin\theta\, d\theta}_{2} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \frac{\hat{r}\cdot\hat{r}}{R^2} R^2 = 4\pi.$$

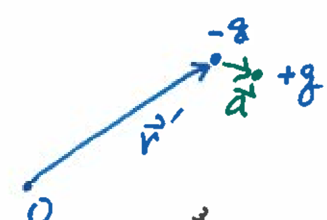
$$\int_V 4\pi\, \delta^3(\vec{r})\, d\tau = 4\pi.$$

EX $\quad J = \int_V (r^2+2)\, \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) d\tau$

$$= \int_V (r^2+2)\, 4\pi\, \delta^3(\vec{r})\, d\tau = 8\pi.$$

EX $\quad$ Consider the following point charge configurations.

(i) Point charge at $\vec{r}'$ $\qquad$ (ii) Dipole at $\vec{r}'$ separated by $\vec{a}$



charge density $\rho(\vec{r}) = q\,\delta^3(\vec{r}-\vec{r}')$

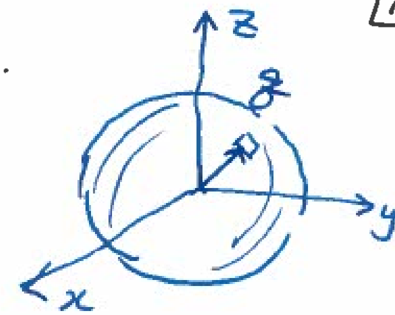
$$\int_V \rho(\vec{r})\, d\tau = \int q\, \delta^3(\vec{r}-\vec{r}')\, d\tau = q$$

$$\rho(\vec{r}) = -q\,\delta^3(\vec{r}-\vec{r}') + q\,\delta^3(\vec{r}-\vec{r}'-\vec{a})$$

$$\int_V \rho(\vec{r})\, d\tau = -q + q = 0.$$

(iii) uniform Surface charge on a sphere of radius R.

$$\rho(\vec{r}) = \frac{q}{4\pi R^2}\underbrace{\delta(r-R)}_{\text{1D delta fct.}}$$



$$\int_V \rho(\vec{r})\, d\tau = 4\pi R^2 \int \frac{q}{4\pi R^2}\delta(\vec{r}-R)\, dr$$

$$= q .$$

$\boxed{\text{HW}}$ Prob. 1.48 (b) & (c).

▸ Delta function & Fourier transformation.

⊙ Let $f(x)$ be a fct. of a real variable x, and as $x \to \pm\infty$ $|x|^n f(x) \to 0$ ~~(bounded)~~ for any $n > 0$. Then we can define the Fourier transform of $f(x)$: f_k

$$f_k = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx \quad \text{and also inversely}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f_k\, e^{+ikx}\, dk .$$

— ①

For a fct in 3D, $f(\vec{r})$, one can define the F.T.

$$f_{\vec{k}} = \frac{1}{(2\pi)^{3/2}}\int d^3r\, f(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}} \quad \text{and} \quad f(\vec{r}) = \frac{1}{(2\pi)^{3/2}}\int d^3k\, f_{\vec{k}}\, e^{+i\vec{k}\cdot\vec{r}}$$

From Eq. ①

$$f_k = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dx\, e^{-ikx} \cdot \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dk'\, f_{k'}\, e^{+ik'x}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk'\, f_{k'} \underbrace{\int_{-\infty}^{+\infty} dx\, e^{+i(k'-k)x}}_{2\pi\delta(k'-k)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk'\, f_{k'}\, 2\pi\,\delta(k'-k) = f_k .$$

Since $\delta(x) = \delta(-x)$ (even function),

$$\int_{-\infty}^{+\infty} dx\, e^{\pm i(k-k')x} = 2\pi\,\delta(k-k') .$$

or
$$\boxed{\int_{-\infty}^{+\infty} dx\, e^{\pm ikx} = 2\pi\,\delta(k) .}$$

Similarly

$$\int_{V_k} e^{i\vec{k}\cdot\vec{r}}\, d^3x = (2\pi)^3 \delta^3(\vec{k})$$

$$\int_V e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} d^3k = (2\pi)^3 \delta^3(\vec{r}-\vec{r}')$$

* Suppose $g(x)$ is a function and $g(x_i) = 0$. Then

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}\delta(x-x_i)$$

e.g. $g(x) = x^2 - 1$, $x_i = \pm 1$

$$\delta(x^2-1) = \sum_i \frac{1}{|g'(x_i)|}\delta(x-x_i) = \frac{1}{2\cdot(1)}\delta(x-1) + \frac{1}{2\cdot(-1)}\delta(x+1)$$

$$= \frac{1}{2}\left(\delta(x-1) - \delta(x+1)\right).$$

$$\therefore \int_{-\infty}^{+\infty} f(x)\,\delta(x^2-1)\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} f(x)\delta(x-1)\,dx - \frac{1}{2}\int_{-\infty}^{+\infty} f(x)\,\delta(x+1)\,dx$$

$$= \frac{1}{2}\left(f(1) - f(-1)\right).$$

EX Using $\int_{-\infty}^{+\infty} e^{-ikx}dx = 2\pi\,\delta(k)$,

$$\int_{-\infty}^{+\infty} x\, e^{-ikx} dx = \frac{1}{(-i)}\int_{-\infty}^{+\infty} \frac{d}{dk}(e^{-ikx})dx = i\frac{d}{dk}\underbrace{\int_{-\infty}^{+\infty} e^{-ikx} dx}_{2\pi\delta(k)}$$

$$= 2\pi i\,\delta'(k) .$$

EX
$$\boxed{\int_{-\infty}^{+\infty} f(x)\delta'(x)\,dx = -f'(0) .}$$

— Integration by part