L3 Ch. 2.

▶ Double derivatives

◎ $\vec{\nabla}\cdot\vec{\nabla}f = \nabla^2 f$ (scalar) "Laplacian"

$$= \partial_i(\vec{\nabla}f)_i = \partial_i(\partial_i f) = \partial_i^2 f$$

$$= (\partial_x^2+\partial_y^2+\partial_z^2)f = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}.$$

$\vec{\nabla}\cdot\vec{\nabla}\vec{A} = \nabla^2\vec{A}$ (vector)

$$= \partial_i\partial_i A_j\hat{e}_j = (\partial_x^2+\partial_y^2+\partial_z^2)A_x\hat{x} + (\partial_x^2+\partial_y^2+\partial_z^2)A_y\hat{y} + \cdots$$

$$= \nabla^2 A_x\hat{x} + \nabla^2 A_y\hat{y} + \nabla^2 A_z\hat{z}.$$

◎ $\vec{\nabla}\times(\vec{\nabla}f) = \epsilon_{ijk}\partial_j(\vec{\nabla}f)_k\hat{e}_i = 0$

$\because \{\vec{\nabla}\times(\vec{\nabla}f)\}_{i=1} = \epsilon_{1jk}\partial_j(\vec{\nabla}f)_k = \epsilon_{1jk}\partial_j\partial_k f$

$$= \epsilon_{123}\partial_2\partial_3 f + \epsilon_{132}\partial_3\partial_2 f$$

$$= \frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y} = 0.$$

Same for other components.

$\vec{\nabla}\times\vec{\nabla} \equiv 0$ (as if $\vec{A}\times\vec{A} = 0$).

○ $\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \partial_i(\partial_j A_j)\hat{e}_i \neq \partial_i\partial_i A_j\hat{e}_j$

○ $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

◎ $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$ —— ①

HW show the above relation ①.

HW Navier-Stokes eq. is the governing eq. of viscous fluid motion:

$$\rho\frac{\partial\vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p + \mu\nabla^2\vec{v}$$

where ρ, $\vec{v}$, P, μ are the density, the velocity, the pressure, and the dynamic viscosity. Write down the full differential eq for the x-component.

Integral Calculus.

- For a scalar field $f(\vec{r})$, there are line, surface, and volume integrals:

$$\int_c f\,dl, \quad \int_s f\,d^2x = \int_s f\,ds, \quad \int_V f\,d^3x = \int_V f\,dV.$$

There are those integrals associated with each component of vector fields $\vec{A}(\vec{r})$:

$$\int_c A_i\,dl, \quad \int_s A_i\,ds, \quad \int_V A_i\,dV$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\int_c \vec{A}\,dl, \quad \int_s \vec{A}\,ds, \quad \int_V \vec{A}\,dV.$$

But there are other types of integrals for a vector field, which are more commonly appearing in physics.

- Line integrals.

$\int_{a\,P}^{b} \vec{A}\cdot d\vec{l}$: integral between two points following a specific path "P".

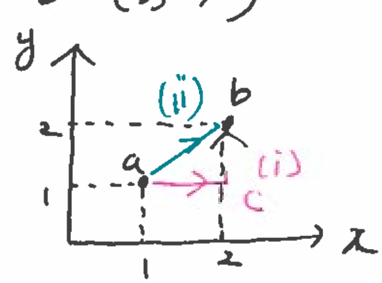
$\oint_c \vec{A}\cdot d\vec{l}$: integral over a specific closed loop "C".

* If $\int_{a\,P_1}^{b} \vec{A}\cdot d\vec{l} = \int_{a\,P_2}^{b} \vec{A}\cdot d\vec{l}$ for an arbitrary paths P_1 & P_2, (independent of path)

then $\vec{A}(\vec{r})$ is a conservative field, and $\oint_c \vec{A}\cdot d\vec{l} = 0$.

Ex $\vec{v} = y^2\hat{x} + 2x(y+1)\hat{y}$; $\vec{a} = (1,1,0)$ & $\vec{b} = (2,2,0)$

$\int_a^b \vec{v}\cdot d\vec{l}$ for paths (i) & (ii)



$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

For path (i)

$$\int_a^b \vec{v}\cdot d\vec{l} = \underbrace{\int_a^c \vec{v}\cdot d\vec{l}}_{y=1\,;\ dy=dz=0} + \underbrace{\int_c^b \vec{v}\cdot d\vec{l}}_{x=2\,;\ dx=dz=0}.$$

$$= \int_1^2 dx\,(1) + \int_1^2 dy\; 4(y+1)$$

$$= 1 + (2y^2+4y)\Big|_1^2 = 1+10 = 11.$$

For Path (ii), $x=y\ ;\ dx=dy\ \&\ dz=0$

$$\int_a^b \vec{v}\cdot d\vec{l} = \int_a^b \{x^2\hat{x} + (2x^2+2x)\hat{y}\}\cdot\{dx\,\hat{x}+dx\,\hat{y}\}$$

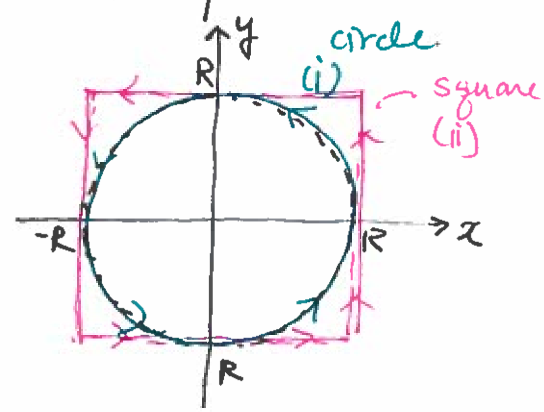
$$= \int_1^2 x^2 dx + \int_1^2 (2x^2+2x)\,dx$$

$$= 10.$$

$\vec{v}(\vec{r})$ is not a conservative field.

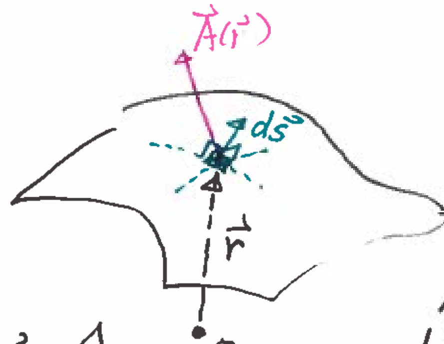
HW $\quad \vec{v}(\vec{r}) = \vec{r}$. Calculate $\oint_c \vec{v}\cdot d\vec{l}$ for paths (i) & (ii)

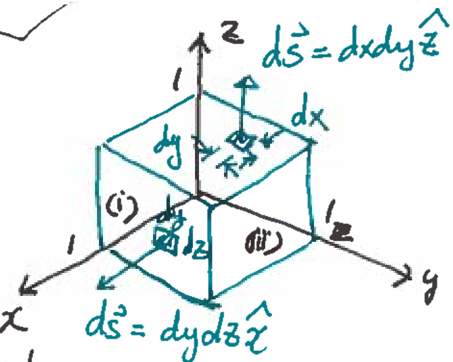
Ans, both "0"



∘ Surface Integral.

$$\underbrace{\int_S \vec{A}\cdot d\vec{s}}_{\text{flux}} \quad \text{or} \quad \oint \vec{A}\cdot d\vec{s}$$





EX $\vec{V} = 2xz\,\hat{x} + (x+2)\hat{y} + y(z^2-3)\hat{z}$

$$\int_{(i)} \vec{V}\cdot d\vec{s} = \int_0^1 dz \int_0^1 dy\,(2z) = 1$$

$x=1$ & $dx=0$
$d\vec{s} = dy\,dz\,\hat{x}$

$$\int_{(iii)} \vec{V}\cdot d\vec{s} = \int_0^1 dx \int_0^1 dz\,(x+2) = \int_0^1 dz\,\left(\tfrac{1}{2}x^2+2x\right)\Big|_0^1$$

$y=1$, $dy=0$
$d\vec{s} = dx\,dz\,\hat{y}$

$$= \int_0^1 \frac{5}{2}\,dz = \frac{5}{2}.$$

⋮

∘ Volume Integral.

▸ Theorems of Stokes and Gauss

$$\int_1^2 \left(\frac{df}{dx}\right) dx = f(x_2) - f(x_1) \quad \text{for a single variable function } f(x).$$

∘ For $F = F(\vec{r})$,

$$dF = \vec{\nabla}F\cdot d\vec{r} = \vec{\nabla}F\cdot d\vec{l}$$

$$\therefore \boxed{\int_1^2 \vec{\nabla}F\cdot d\vec{l} = F(\vec{r}_2) - F(\vec{r}_1)}$$ "fundamental theorem of gradient"

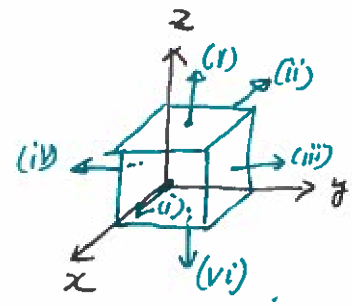
In other words, if a vector field $\vec{A}(\vec{r}) = \vec{\nabla}F$, then $\int_1^2 \vec{A}\cdot d\vec{l}$ is independent of the path between two end pts.

conservative vector field.

- Gauss's theorem or Divergence theorem.

$$\boxed{\underbrace{\int_V (\vec{\nabla}\cdot\vec{A})\,dV}_{\text{Volume Int.}} = \underbrace{\oint_S \vec{A}\cdot d\vec{s}}_{\text{Surface int.}}}$$

Ex $\vec{V} = y^2\hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}$



$$\vec{\nabla}\cdot\vec{V} = 2x + 2y = 2(x+y).$$

$$\int_V (\vec{\nabla}\cdot\vec{V})\,dV = \int_0^1 dx\int_0^1 dy\int_0^1 dz\, 2(x+y).$$

$$= 2\int_0^1 dx\int_0^1 dy\,(x+y)z\Big|_{z=0}^{z=1} = 2\int_0^1 dx\int_0^1 dy\,(x+y)$$

$$= 2\int_0^1 dx\,\left(xy + \tfrac{1}{2}y^2\right)\Big|_{y=0}^{y=1}$$

$$= 2\int_0^1 \left(x + \tfrac{1}{2}\right)dx = x^2 + x\Big|_0^1 = 2.$$

$$\oint_S \vec{V}\cdot d\vec{s} = \left(\int_{(i)} + \int_{(ii)} + \int_{(iii)} + \int + \cdots + \int_{(vi)}\right)\vec{V}\cdot d\vec{s}.$$

$$\int_{(i)\,(x=1)} \vec{V}\cdot d\vec{s} = \int_0^1\int_0^1 dy\,dz\cdot y^2 = \frac{1}{3}.$$

$$\int_{(ii)\,(x=0)} \vec{V}\cdot d\vec{s} = \int_0^1\int_0^1 dy\,dz\,(-y^2) = -\frac{1}{3}$$

$$\int_{(iii)\,(y=1)} \vec{V}\cdot d\vec{s} = \int_0^1\int_0^1 dx\,dz\,(2x\cancel{y} + z^2) = \int_0^1 dx\,\left(2x + \tfrac{1}{3}\right) = \frac{4}{3}$$

$$\int_{(iv)\,(y=0)} \vec{V}\cdot d\vec{s} = -\int_0^1\int_0^1 dx\,dz\,(0 + z^2) = -\frac{1}{3}$$

$$\int_{(v)\,(z=1)} \vec{v}\cdot d\vec{s} = \int_0^1\!\!\int_0^1 dx\,dy\,(2y) = 1$$

$$\int_{(vi)\,(z=0)} \vec{v}\cdot d\vec{s} = 0 .$$

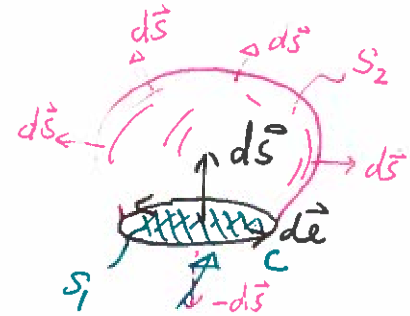
$$\therefore\ (i)+(ii)+\cdots+(vi) = \frac{1}{3}-\frac{1}{3}+\frac{4}{3}-\frac{1}{3}+1+0 = 2 .$$

$$\therefore \int_V (\vec{\nabla}\cdot\vec{v})\,dV = \oint_S \vec{v}\cdot d\vec{s} .$$

⑥ Stoke's theorem

$$\boxed{\int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{s} = \oint_C \vec{A}\cdot d\vec{l}}$$

→ Right-Hand-Rule Convention



$$\int_{S_1} (\vec{\nabla}\times\vec{A})\cdot d\vec{s} = \int_{S_2} (\vec{\nabla}\times\vec{A})\cdot d\vec{s}$$ (surface S_1 & S_2 are bound by the closed loop C!).

(Proof)

$$\int_{S_1} (\vec{\nabla}\times\vec{A})\cdot d\vec{s} - \int_{S_2} (\vec{\nabla}\times\vec{A})\cdot d\vec{s} = \oint_{S=S_1+S_2} (\vec{\nabla}\times\vec{A})\cdot d\vec{s}$$

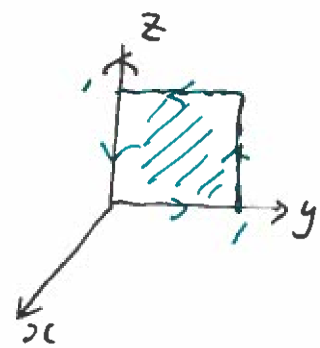
↑ Gauss' theorem

$$= \int_V \underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})}_{=0}\,dV$$

$$= 0 .$$

Ex $\vec{v} = (2xz+3y^2)\,\hat{y} + 4yz^2\,\hat{z}$

$$\int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{s} = \oint \vec{v}\cdot d\vec{l}$$



HW Prob. 1.33 & 1.34

- Few useful relations & theorems
  - $\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f)$

$$\int_V \vec{\nabla}\cdot(f\vec{A})dV = \int_V f(\vec{\nabla}\cdot\vec{A})dV + \int_V \vec{A}\cdot(\vec{\nabla}f)\cdot dV = \oint f\vec{A}\cdot d\vec{l}$$

$$\therefore \int_V f(\vec{\nabla}\cdot\vec{A})dV = -\int_V \vec{A}\cdot(\vec{\nabla}f)dV + \oint f\vec{A}\cdot d\vec{l}$$

$\boxed{\text{HW}}$ Prob 1.36

(a) use stokes theorem (b) use divergence theorem.

- Green's Identities.

  Two scalar fields $F_1(\vec{r})$ & $F_2(\vec{r})$.

$$\int_V (F_1\nabla^2F_2 + \vec{\nabla}F_1\cdot\vec{\nabla}F_2)dV = \oint_S F_1\vec{\nabla}F_2\cdot d\vec{s}$$

$$\int_V (F_1\nabla^2F_2 - F_2\nabla^2F_1)dV = \oint_S (F_1\vec{\nabla}F_2 - F_2\vec{\nabla}F_1)\cdot d\vec{s}$$

- Both Gauss's and Stokes's theorem map d-dimensional integral onto (d-1)-dimensional integral.

$$\int_V \vec{\nabla}\cdot\vec{V}\,dV = \oint_S \vec{V}\cdot d\vec{s} \qquad 3d \to 2d$$

$$\int_S (\vec{\nabla}\times\vec{V})\cdot d\vec{s} = \oint_C \vec{V}\cdot d\vec{l} \qquad 2d \to 1d$$

In general,

$$\int_V \vec{\nabla}\,dV \longrightarrow \oint_S \hat{n}\,ds \quad (\hat{n}: \text{unit vector of } d\vec{s})$$

$$\int_S (d\vec{s}\times\vec{\nabla}) \longrightarrow \oint_C d\vec{l}$$

e.g.

$$\int_S (d\vec{s}\times\vec{\nabla})\times\vec{V} = \oint_C (d\vec{l}\times\vec{V})$$

$$\int_V \vec{\nabla}f\,dV = \oint_S \hat{n}f\,ds = \oint_S f\,d\vec{s}$$

HW * Verify Stokes thm for $\vec{A} = (x+y)\hat{x} - 2x^2\hat{y} + xy\hat{z}$ and the upper hemisphere of ~~rad~~ $R=1$; $x^2+y^2+z^2=1$

$$\oint_S (\nabla \times \vec{A})\cdot d\vec{s} = \oint_C \vec{A}\, d\vec{l}$$

Try this but you'll find this is much easier!