$\boxed{L21}$ Magnetic Fields in Matter

▸ Magnetization.

Electric Dipole

point-like dipole moment

$$\vec{p} = \int \vec{r}' \rho(\vec{r}') d\tau'$$

potential from $\vec{p}$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2}$$

($\vec{p}$ at the origin)

$$\vec{E}_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$

$$= \underbrace{\frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)}_{\text{when } \vec{p} = p\hat{z}}$$

$$U_E = -\vec{p}\cdot\vec{E}$$

$$\vec{F}_p = -\vec{\nabla} U_E = \vec{\nabla}(\vec{p}\cdot\vec{E}) = (\vec{p}\cdot\vec{\nabla})\vec{E}$$

$$\vec{N}_p = \vec{p}\times\vec{E}$$

$\vec{P}$: dipole density in dielectric mat

polarization

Magnetic Dipole

point-like dipole moment.

$$\vec{m} = I\int_S d\vec{a}'$$

Vector potential from $\vec{m}$

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

($\vec{m}$ at the origin).

$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{p}\right].$$

$$= \underbrace{\frac{\mu_0 m}{4\pi r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]}_{\text{when } \vec{m} = m\hat{z}}.$$

$$U_m = -\vec{m}\cdot\vec{B}.$$

$$\vec{F}_m = -\vec{\nabla} U_m = \vec{\nabla}(\vec{m}\cdot\vec{B}) = (\vec{m}\cdot\vec{\nabla})\vec{B}$$

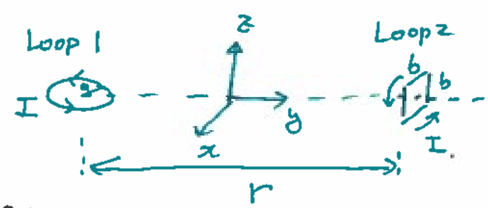
$$\vec{N}_m = \vec{m}\times\vec{B}$$

$\vec{M}$: magnetic dipole density in mag. material

magnetization.

$\boxed{HW}$ 6.3, 6.5

EX Torque on L2 by L1.



$$\vec{m}_1 = \pi a^2 I\,\hat{z}$$

$$\vec{m}_2 = b^2 I\,\hat{y}.$$

Field produced by $\vec{m}_1$ at $\vec{r} = r\hat{y}$: $\vec{B}_1(\vec{r})$.

$$\vec{B}_1(\vec{r}) = \frac{\mu_0}{4\pi}\cdot\frac{1}{r^3}\left[3(\vec{m}_1\cdot\hat{r})\hat{r} - \vec{m}_1\right]$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3m_1\cancel{(\hat{z}\cdot\hat{y})}^{\,0}\hat{y} - m_1\hat{z}\right] = -\frac{\mu_0 m_1}{4\pi r^3}\hat{z}$$

$$\therefore \ \vec{N}_2 = \vec{m}_2 \times \vec{B}_1 = -\frac{\mu_0 m_1 m_2}{4\pi r^3}(\hat{y}\times\hat{z}) = -\frac{\mu_0 m_1 m_2}{4\pi r^3}\hat{x}$$

where $m_1 = \pi a^2 I$ and $m_2 = b^2 I$.

$\vec{N}_2$ will rotate $\vec{m}_2$ around $\hat{x}$ axis and align $\vec{m}_2$ to $(-\hat{z})$.

Unlike $\vec{P}$ (polarization), $\vec{M}$ (magnetization) can be induced in the same direction as the magnetic field and also in the opposite direction to the magnetic field.

paramagnetism

Diamagnetism.

- Fields from a magnetized object.

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{\mathscr{r}}}{\mathscr{r}^2} \Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \Rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\vec{P}(\vec{r}')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

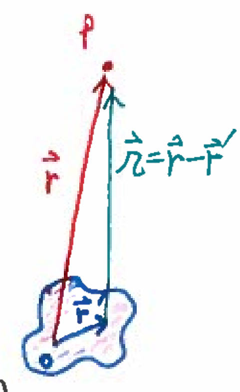
$$= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_b(\vec{r}')}{\mathscr{r}^2}d\tau' + \frac{1}{4\pi\varepsilon_0}\oint \frac{\sigma_b(\vec{r}')}{\mathscr{r}^2}da'$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau' = \frac{\mu_0}{4\pi}\int \vec{M}(\vec{r}')\times\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right)d\tau'$$

$$\left(\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

$$= \frac{\mu_0}{4\pi}\left[\int \frac{1}{\mathscr{r}}[\vec{\nabla}'\times\vec{M}(\vec{r}')]d\tau' - \int \vec{\nabla}'\times\left(\frac{\vec{M}(\vec{r}')}{\mathscr{r}}\right)d\tau'\right]$$

$$= \frac{\mu_0}{4\pi}\left[\int \frac{1}{\mathscr{r}}(\vec{\nabla}'\times\vec{M}(\vec{r}'))d\tau' + \oint_S \frac{1}{\mathscr{r}}(\vec{M}(\vec{r}')\times d\vec{a}')\right]$$



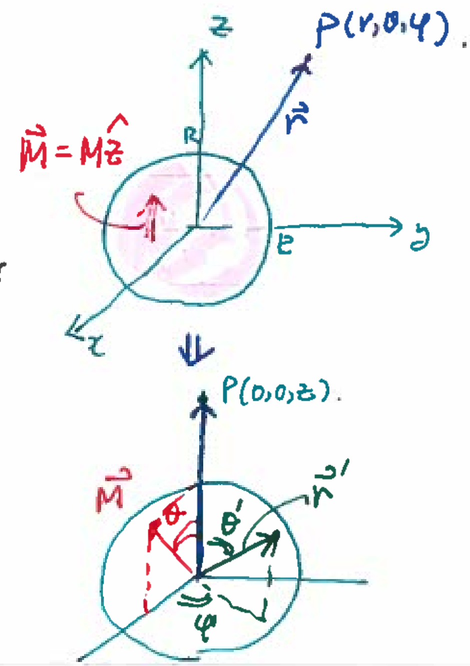
We can define new quantities

$\vec{J}_b = \vec{\nabla}\times\vec{M}$: bound vol. current density

$\vec{K}_b = \vec{M}\times\hat{n}$: bound surf. current density.

$$\boxed{\therefore\ \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}\,d\tau' + \frac{\mu_0}{4\pi}\oint_S \frac{\vec{K}(\vec{r}\,')}{\mathscr{r}}\,da'}$$

<u>EX</u>. Uniformly magnetized sphere



$$\vec{J}_b = \vec{\nabla}\times\vec{M} = 0$$

$$\vec{K}_b = \vec{M}\times\hat{n} = \vec{M}\times\hat{n} = M\sin\theta\,\hat{\varphi}$$

As we did in EX 5.11, align P along the z-axis. Then.

$$\vec{M} = M\sin\theta\,\hat{x} + M\cos\theta\,\hat{z}$$

$$\vec{r}\,' = R(\sin\theta'\sin\varphi'\hat{x} + \sin\theta'\cos\varphi'\hat{y} + \cos\theta'\hat{z})$$

$$\therefore\ \vec{K}_b(\vec{r}) = \vec{M}\times\vec{r}\,'$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ M\sin\theta & 0 & M\cos\theta \\ R\sin\theta'\sin\varphi' & \sin\theta'\cos\varphi' & \cos\theta' \end{vmatrix}$$

$$= RM\left[-(\cos\theta\sin\theta'\cancel{\cos\varphi'})\hat{x} + (\cos\theta\sin\theta'\cancel{\cos\varphi'} - \sin\theta\cos\theta')\hat{y} + \sin\theta\sin\theta'\cancel{\sin\varphi'}\,\hat{z}\right]$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' \cancel{= z\hat{z} - R(\sin\theta'\sin}$$

$$\therefore\ \mathscr{r} = \sqrt{r^2 + R^2 - 2rR\cos\theta'}$$

$$\therefore\ \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_0^{2\pi}\!\!\int_0^{\pi} R^2\sin\theta'\,d\theta'\,d\varphi'\cdot\frac{\vec{K}_b(\vec{r}\,')}{\mathscr{r}}$$

$$= \frac{\mu_0}{4\pi}R^2(-RM)(2\pi)\,\sin\theta\,\underbrace{\int_0^{\pi}\frac{\cos\theta'\sin\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\,d\theta'}_{\frac{2R}{3r^2}\text{ for } r>R.}\,\hat{y}$$

$$= -\frac{\mu_0 R^4 M}{3r^2}\hat{y} = \frac{\mu_0 R^4 M}{3r^2}(\hat{M}\times\hat{r})$$

Going back to the original coordinate.

$$\vec{A}(r,\theta,\varphi) = \frac{\mu_0 R^4 M}{3}\frac{\sin\theta}{r^2}\hat{\varphi}. = A_\varphi\hat{\varphi}.$$

$$\therefore \vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\varphi\right)\hat{r} - \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left(r\sin\theta A_\varphi\right)\hat{\theta}$$

$$= \frac{\mu_0 R^4 M}{3}\left[\frac{1}{r^3\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)\hat{r} - \frac{\sin^2\theta}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{\theta}\right]$$

$$= \frac{\mu_0 R^4 M}{3}\cdot\frac{1}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

$$\left(m = \frac{4}{3}\pi R^3 M\right) \quad = \frac{\mu_0 m}{4\pi}\cdot\frac{1}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right].$$

Finish the calculation for $r<R$, to the final answer $\vec{B} = \frac{2}{3}\mu_0\vec{M}$.

$$\int_0^\pi \frac{\cos\theta'\sin\theta'}{\sqrt{\quad}}d\theta' = \frac{2r}{3R^2}.$$

- $\vec{H}$

Remember how we introduced $\vec{D}$?

Starting from $\vec{E}$,

$$\vec{\nabla}\cdot\vec{E} = \rho_{en}/\varepsilon_0 = \frac{1}{\varepsilon_0}(\rho_f + \rho_b)$$: all sources contribute to $\vec{E}$!

$$= \frac{1}{\varepsilon_0}(\rho_f - \vec{\nabla}\cdot\vec{P})$$: $\vec{P}$ produces bound charge ρ_b.

$$\therefore \vec{\nabla}\cdot\left(\vec{E} + \frac{1}{\varepsilon_0}\vec{P}\right) = \frac{1}{\varepsilon_0}\vec{\nabla}\cdot\vec{D} = \frac{\rho_f}{\varepsilon_0}.$$

$$\boxed{\vec{D} = \varepsilon_0\vec{E} + \vec{P}} \qquad \vec{\nabla}\cdot\vec{D} = \rho_f$$: Only free charge contributes to $\vec{D}$

Now, starting from Ampere's law

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}_{en} = \mu_0(\vec{J}_f + \vec{J}_b)$$: all sources contribute to $\vec{B}$

$$= \mu_0(\vec{J}_f + \vec{\nabla}\times\vec{M})$$: $\vec{M}$ produces bound current $\vec{J}_b$

$$\therefore \vec{\nabla}\times(\vec{B} - \mu_0\vec{M}) = \mu_0\vec{J}_f$$

Introduce an auxiliary field $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$, : called H-field

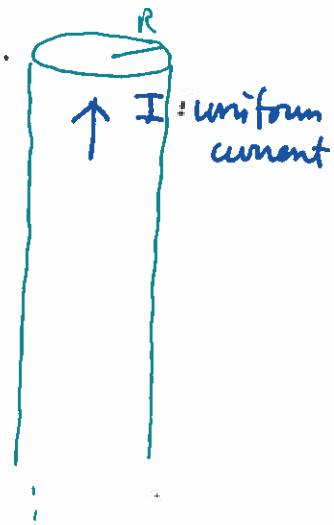
$$\vec{\nabla}\times\vec{H} = \vec{J}_f$$: Only free current contributes to $\vec{H}$!

No μ_0 here!

Therefore

$$\oint \vec{H}\cdot d\vec{l} = (I_f)_{en}$$: Ampere's law for $\vec{H}$

<ins>EX</ins> Uniform current I through a cylindrical wire. $\vec{H}$ for $r<R$ and $r>R$.



$I = I_f$: free current

$$J_f = \frac{I}{\pi R^2}.$$

$\therefore$ For $r<R$, applying Ampere's law since $\vec{H}\propto\hat{\varphi}$

$$2\pi r H_\varphi = \pi r^2 J_f = \pi r^2 \frac{I}{\pi R^2} = I\frac{r^2}{R^2}$$

$$\therefore H_\varphi = \frac{I}{2\pi}\frac{r}{R^2} \Rightarrow \vec{H} = \frac{I}{2\pi R^2} r\,\hat{\varphi}.$$

For $r>R$

$$2\pi r H_\varphi = I \Rightarrow \vec{H} = \frac{I}{2\pi r}\hat{\varphi}$$

Here $\vec{M}=0$ (vacuum), $\therefore \vec{B} = \mu_0\vec{H} = \frac{\mu_0 I}{2\pi r}\hat{\varphi}$

$\boxed{\text{HW}}$ 6.12, 6.13.

- Boundary Conditions

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{\nabla}\cdot(\vec{H}+\vec{M}) = 0.$$

Therefore

$$\boxed{H^\perp_{above} - H^\perp_{below} = -\left(M^\perp_{above} - M^\perp_{below}\right)}$$

And $\vec{\nabla}\times\vec{H} = \vec{J}_f$,

$$\boxed{\vec{H}^\parallel_{above} - \vec{H}^\parallel_{below} = \vec{K}_f \times \hat{n}}$$