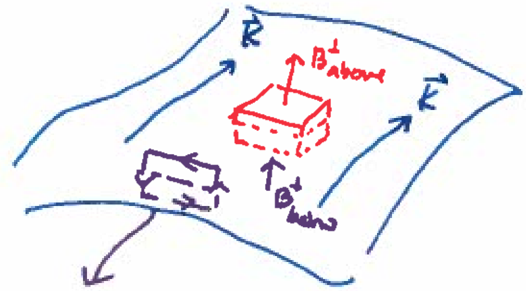
L20. Boundary Value Problem in Magnetostatics

$$\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \boxed{B^{\perp}_{above} = B^{\perp}_{below}}$$

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J} \Rightarrow \boxed{B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K}$$

Component of B parallel to the surface but perpendicular to $\vec{K}$.



The orientation of the loop is ~~perp~~ in the same direction of $\vec{K}$.
Therefore the loop is perpendicular to $\vec{K}$.

But if the loop is aligned perpendicular to $\vec{K}$, then

$$B^{\parallel}_{above} - B^{\parallel}_{below} = 0$$

no current through the surface.

Combined,

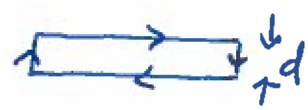
$$\boxed{\vec{B}_{above} - \vec{B}_{below} = \mu_0(\vec{K}\times\underbrace{\hat{n}}_{\text{surface normal vector}})}$$

And,

$$\boxed{\vec{A}_{above} = \vec{A}_{below}}$$

$\because$ Since $\vec{\nabla}\cdot\vec{A}=0$, $A^{\perp}$ should continuous. and.

$$\oint_C \vec{A}\cdot d\vec{l} = \int_S(\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \int_S \vec{B}\cdot d\vec{a} = \Phi_m .$$



By making $d\to 0$, $\Phi_m \to 0$. Therefore $A^{\parallel}$ is also continuous.

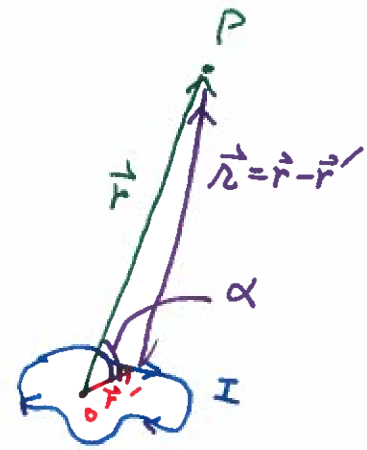
However,

$$\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0\vec{K}$$

- Multipole Expansion of $\vec{A}(\vec{r})$.

Evaluating $\vec{A}(\vec{r})$ from a localized current for $r >> r'$

$$\frac{1}{\eta} = \frac{1}{\sqrt{r^2+(r')^2-2rr'\cos\alpha}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha).$$



$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{d\vec{l}'}{\eta}$$

$$= \frac{\mu_0 I}{4\pi}\oint \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)\, d\vec{l}'$$

$$= \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\alpha)\, d\vec{l}'$$

$$= \frac{\mu_0 I}{4\pi}\Big[\underbrace{\frac{1}{r}\oint P_0(\cos\alpha)\, d\vec{l}'}_{\text{monopole}} + \underbrace{\frac{1}{r^2}\oint r' P_1(\cos\alpha)\, d\vec{l}'}_{\text{dipole}} + \underbrace{\frac{1}{r^3}\oint (r')^2 P_2(\cos\alpha)\, d\vec{l}'}_{\text{quadrupole}} + \cdots\Big]$$

M: $\oint P_0(\cos\alpha)\, d\vec{l}' = \oint d\vec{l}' = 0 \Rightarrow$ No monopole $(\vec{\nabla}\cdot\vec{B} = 0)$

D: $\oint r' P_1(\cos\alpha)\, d\vec{l}' = \oint r'\cos\alpha\, d\vec{l}' = \oint (\underbrace{\hat{r}}_{\text{const vector}}\cdot\vec{r}')\, d\vec{l}' = -\hat{r}\times\int_S d\vec{a}'$

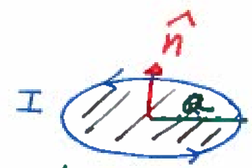
$$\left[\oint_C (\vec{c}\cdot\vec{r})\, d\vec{l} = \left(\frac{1}{2}\oint_C \vec{r}\times d\vec{l}\right)\times\vec{c} \quad \text{for any const. } \vec{c}\right].$$

$$= \int_S d\vec{a}\times\vec{c}$$

$$\therefore \vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{I}{r^2}\int_S d\vec{a}'\times\hat{r} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

where $\boxed{\vec{m} = I\int d\vec{a}'}$ Magnetic dipole moment.

$$\vec{m} = I \int_S d\vec{a}\,' = \pi a^2 I \,\hat{n}$$

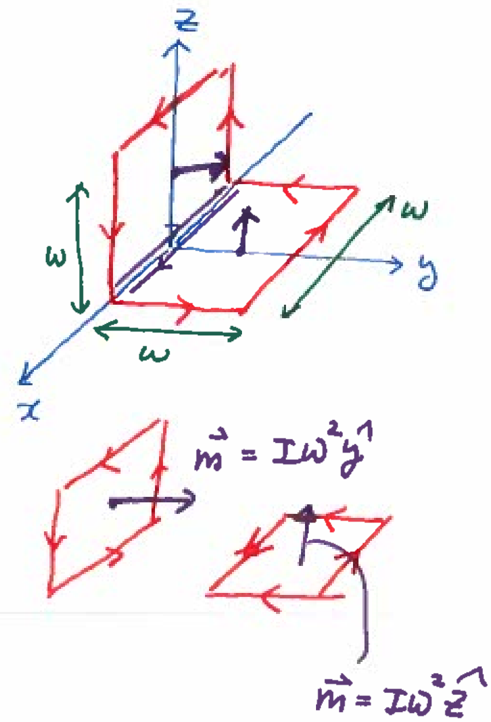


magnetic dipole moment of a circular loop current.

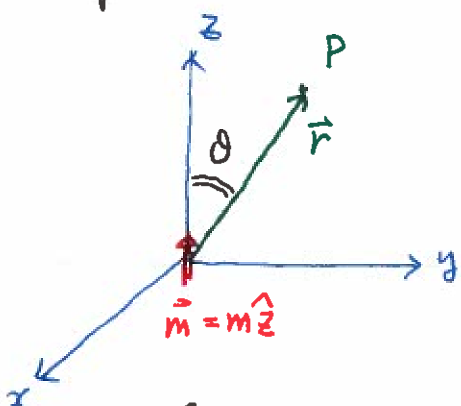
EX

Split into two loops

$$\vec{m} = I w^2 (\hat{y} + \hat{z}).$$



- $\vec{B}$ from a dipole moment $\vec{m}$



$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi} \cdot \frac{\vec{m} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} \frac{m \sin\theta}{r^2} \hat{\varphi}$$

$$\vec{B}_{dip}(\vec{r}) = \vec{\nabla} \times \vec{A} = \frac{\mu_0 m}{4\pi} \frac{1}{r^3} \left(2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta} \right).$$

$\boxed{\text{HW}}$ 5.36, 5.37

$$\vec{B}_{dip} = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m} \right] \qquad \vec{m} = I \int_S d\vec{a}$$

$$\vec{E}_{dip} = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^3} \left[3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p} \right] \qquad \vec{p} = \int \vec{r}\,' \rho(\vec{r}\,') d\tau'$$