L2 Ch. 1. Vector Analysis (Differential Calculus) Lt

- Scalar fields & Vector fields.

$f(x,y,z,t)$ $\vec{A}(x,y,z,t)$.

At each point in space and at a given time, a scalar quantity is assigned: → Ignore the time dependence for a while ⇒ Static scalar / vector fields.

- temperature distribution in a room. (3D)
- altitude of landscape (2D)
- ...

At each point in space and at a given time, a vector ~~quantity~~ quantity is assigned: flow velocity in a river.

- Derivatives of vector fields.

⊙ If $f = f(x)$ (function of x; 1D scalar fields), then one can define

$$\left.\frac{df}{dx}\right|_{x_0} = \lim_{\varepsilon\to 0} \frac{f(x_0+\varepsilon)-f(x_0)}{\varepsilon}$$

⊙ If $\vec{v} = \vec{v}(t)$ (~~function~~ a time-dep. vector), then one can define

$$\frac{d\vec{v}}{dt} = \lim_{\Delta t\to 0} \frac{\vec{v}(t+\Delta t)-\vec{v}(t)}{\Delta t}.$$

In other words, $\left.\frac{df}{dx}\right|_{x_0}$ would tell us how does f varies at x_0 when the argument deviates from x_0 by an infinitesimal amount dx:

$$df = \left(\frac{df}{dx}\right) dx.$$

← Linear relation

⊙ Extending to 3D scalar fields, $F(x,y,z)$. → partial derivatives!

$$dF = \left(\frac{\partial F}{\partial x}\right)_{y,z} dx + \left(\frac{\partial F}{\partial y}\right)_{x,z} dy + \left(\frac{\partial F}{\partial z}\right)_{x,y} dz$$

In a Cartesian coordinates, one can express the above in a more compact form:

$$dF = \underbrace{\left(\frac{\partial F}{\partial x}\hat{x} + \frac{\partial F}{\partial y}\hat{y} + \frac{\partial F}{\partial z}\hat{z}\right)}_{\equiv\ \vec{\nabla}F\ :\ \text{gradient of } F} \cdot \big(\underbrace{dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}}_{=\,d\vec{r}}\big)$$

$$= \underbrace{\vec{\nabla}F}_{\text{vector fields}} \cdot \underbrace{d\vec{r}}_{\text{vector fields}} .$$

$$= |\vec{\nabla}F|\,|d\vec{r}|\cos\theta .$$

* dF is maximum when $\theta = 0$. i.e. $\vec{\nabla}F$ is in the same direction as $d\vec{r}$.

** Therefore, $\vec{\nabla}F$ indicates the direction of maximum increase in F. And $|\vec{\nabla}F|$ is the slope of change in that direction.

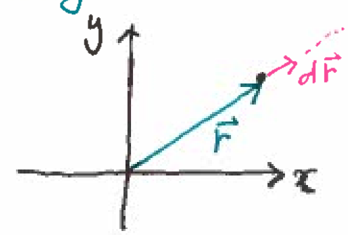
Ex $r = \sqrt{x^2+y^2+z^2}$; $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$.

$$\vec{\nabla}r = \left(\frac{\partial r}{\partial x}\right)\hat{x} + \left(\frac{\partial r}{\partial y}\right)\hat{y} + \left(\frac{\partial r}{\partial z}\right)\hat{z} .$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}x}{\sqrt{x^2+y^2+z^2}}\hat{x} + \cdots + \frac{1}{\cancel{2}}\frac{\cancel{2}z}{\sqrt{x^2+y^2+z^2}}\hat{z}$$

$$= \frac{\vec{r}}{r} = \hat{r}$$

r : distance from the origin changes maximally in the radial dir.



HW Prob. 1.12 & 1.13 & ~~1.14~~

geometrical meaning of $\vec{\nabla}$ → $\vec{\nabla}(r^2)$ & $\vec{\nabla}(\frac{1}{r})$

▸ "Del" operator.

$$\vec{\nabla}F = \left(\frac{\partial F}{\partial x}\right)\hat{x} + \left(\frac{\partial F}{\partial y}\right)\hat{y} + \left(\frac{\partial F}{\partial z}\right)\hat{z}$$

$$= \Big(\underbrace{\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}}\Big)F$$

operator acting on F (everything following $\vec{\nabla}$).

$$\boxed{\begin{aligned}\vec{\nabla} &= \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \\ &= \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z \\ &= \hat{e}_i\partial_i\end{aligned}}$$

"Vector operator".
behaves as if it is a vector!!

◦ Three basic ways of operations (action)

(i) $\vec{\nabla} F(x,y,z) = (\hat{e}_i\partial_i)F = \hat{x}\left(\frac{\partial F}{\partial x}\right) + \hat{y}\left(\frac{\partial F}{\partial y}\right) + \hat{z}\left(\frac{\partial F}{\partial z}\right)$ ← vector.
"Gradient"

(ii) $\vec{\nabla}\cdot\vec{A}(x,y,z) = (\hat{e}_i\partial_i)\cdot(\hat{e}_j A_j) = (\hat{e}_i\cdot\hat{e}_j)\,\partial_i A_j = \delta_{ij}\partial_i A_j$
"Divergence"

operates on all but $\hat{e}_j$ is fixed!

$$= \partial_i A_i = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

↑ Scalar

(iii) $\vec{\nabla}\times\vec{A}(x,y,z)$ "curl"

$= \cancel{(\hat{e}_i\partial_i)\times(\hat{e}_j A_j)}$

special attention for this

$$\vec{\nabla}\times\vec{A} = \epsilon_{ijk}\,\partial_j A_k\,\hat{e}_i$$ ← vector

$$= \hat{e}_1(\partial_2 A_3 - \partial_3 A_2) + \hat{e}_2(\partial_3 A_1 - \partial_1 A_3) + \hat{e}_3(\partial_1 A_2 - \partial_2 A_1)$$

$$= \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$$

—— skip ——

EX Recognizing $\vec{\nabla}$ as a vector operator,

(i) $\vec{\nabla}\cdot\vec{\nabla}$(scalar field) → scalar

(ii) $\vec{\nabla}\times\vec{\nabla}$(scalar field) $= 0$ $\quad (\vec{A}\times\vec{A} = 0)$

(iii) $\vec{\nabla}\cdot(\vec{\nabla}\times$(vector field)$) = 0$ $\quad (\vec{A}\cdot(\vec{A}\times\vec{B}) = 0)$.

(iv) $\vec{\nabla}(\underbrace{\vec{\nabla}\cdot(\text{vector field})}_{\text{scalar field}})$ → vector

(v) $\vec{\nabla}\times(\vec{\nabla}\times$ vector field$)$ → vector.

Divergence

$$\vec{\nabla}\cdot\vec{A} = \partial_i A_i = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

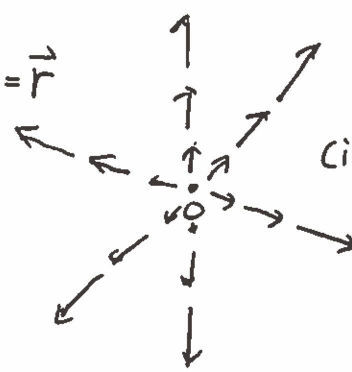
* geometrically $\vec{\nabla}\cdot\vec{A}$ is a measure of how much the vector $\vec{A}$ spreads out (diverges) from the point of interest.
local

** Consider the vector field $\vec{A}$ as flow velocity, then a point of $\vec{\nabla}\cdot\vec{A} > 0\,(<0)$ is a source (sink).

EX

(i) Consider $\vec{A}(x,y,z) = x\hat{x} + y\hat{y} + z\hat{z} = \vec{r}$

$$\vec{\nabla}\cdot\vec{A} = \partial_x x + \partial_y y + \partial_z z = +3$$



(i)

(ii) $\vec{A} = a\hat{z}$

$$\vec{\nabla}\cdot\vec{A} = \partial_z a = 0$$

(ii)

(iii) $\vec{A} = z\hat{z}$

$$\vec{\nabla}\cdot\vec{A} = \partial_z z = +1$$

(iii)

(iv) $\vec{A} = y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}$

$$\begin{aligned}\vec{\nabla}\cdot\vec{A} &= \partial_x y^2 + \partial_y(2xy+z^2) + \partial_z(2yz)\\ &= 0 + 2x + 2y\\ &= 2(x+y)\end{aligned}$$

HW Prob. 1.16

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = ?$$

▷ Curl

$$\vec{\nabla}\times\vec{A} = \epsilon_{ijk}\,\partial_j A_k\,\hat{e}_i$$

* Geometrically $\vec{\nabla}\times\vec{A}$ is a measure of how much the vector $\vec{A}$ swirls around the point of interest.

** Imagine placing a paddlewheel at the point. If the vector fields make the wheel rotate, then $\vec{\nabla}\times\vec{A}\neq 0$

*** The direction of $\vec{\nabla}\times\vec{A}$ indicates the direction of the rotation axis.

EX

(i) $\vec{A} = \omega(-y\hat{x} + x\hat{y})$ $(\omega > 0)$.

$$\vec{\nabla}\times\vec{A} = \omega\left\{ (\partial_y V_z - \cancelto{0}{\partial_z V_y})\hat{x} + (\cancelto{0}{\partial_z V_x - \partial_x V_z})\hat{y} + (-\partial_x V_y + \partial_y V_x)\hat{z} \right\}$$

$$= \underbrace{2\omega}_{\text{everywhere}}\hat{z}\ (>0).$$

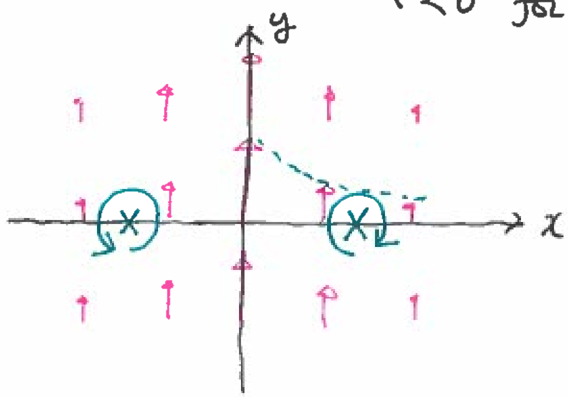
(ii) $\vec{A} = e^{-y^2}\hat{y}$

$$\vec{\nabla}\times\vec{A} = (\partial_y A_z - \partial_z A_y)\hat{x} + (\partial_x A_y - \partial_y A_z)\hat{z}$$

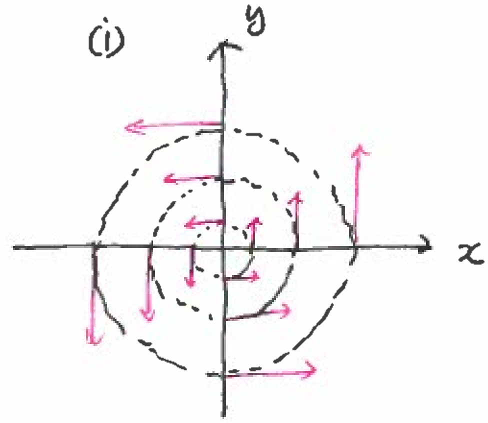
$$= 0$$

(iii) $\vec{A} = e^{-x^2}\hat{y}$

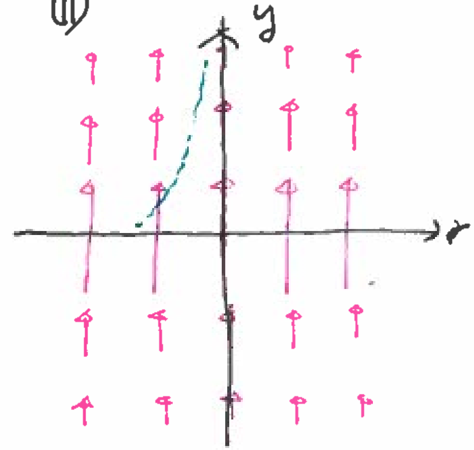
$$\vec{\nabla}\times\vec{A} = -2x e^{-x^2}\hat{z} = \begin{cases} >0 & \text{for } x<0 \\ <0 & \text{for } x>0 \end{cases}$$



(i)



(ii)



HW Prob. 1.18 .

▶ Product rules ~~and double derivatives~~ :

○ Ordinary derivatives

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

$$\frac{d}{dx}(Kf) = K\frac{df}{dx} \quad \text{for } K : \text{const} .$$

$$\frac{d}{dx}(f \cdot g) = f\frac{dg}{dx} + g\frac{df}{dx}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} .$$

○ $\vec{\nabla}(\alpha f + \beta g) = \alpha\vec{\nabla} f + \beta\vec{\nabla} g$; $\vec{\nabla}\cdot(\alpha\vec{A} + \beta\vec{B}) = \alpha\vec{\nabla}\cdot\vec{A} + \beta\vec{\nabla}\cdot\vec{B}$

$\vec{\nabla}\times(\alpha\vec{A} + \beta\vec{B}) = \alpha\vec{\nabla}\times\vec{A} + \beta\vec{\nabla}\times\vec{B}$

where α, β are const.

$$\vec{\nabla}(fg) = f\vec{\nabla}g + g\vec{\nabla}f$$

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}$$

$+$ $\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot\vec{\nabla}f$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

$*$ $\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\vec{\nabla}f)$.

$*$ $\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})$.

$$\vec{\nabla}\left(\frac{f}{g}\right) = \frac{g\vec{\nabla}f - f\vec{\nabla}g}{g^2}$$

$$\vec{\nabla}\cdot\left(\frac{\vec{A}}{g}\right) = \frac{g\vec{\nabla}\cdot\vec{A} - \vec{A}\cdot(\vec{\nabla}g)}{g^2}$$

$$\vec{\nabla}\times\left(\frac{\vec{A}}{g}\right) = \frac{g(\vec{\nabla}\times\vec{A}) + \vec{A}\times(\vec{\nabla}g)}{g^2}$$

$+$ $\vec{\nabla}\cdot(fA) = \partial_i(fA_i) = A_i\partial_i f + f\partial_i A_i = \vec{A}\cdot\vec{\nabla}f + f\vec{\nabla}\cdot\vec{A}$

$$\vec{\nabla}\times(f\vec{A}) = \epsilon_{ijk}\,\partial_j(fA_k)\,\hat{e}_i$$
$$= \epsilon_{ijk}\{A_k(\partial_j f) + f(\partial_j A_k)\}\hat{e}_i$$
$$= \epsilon_{ijk}(\partial_j f)A_k\,\hat{e}_i + f\,\epsilon_{ijk}\partial_j A_k\,\hat{e}_i$$
$$= \vec{\nabla}f\times\vec{A} + f\,\vec{\nabla}\times\vec{A}$$

*. R. Feynman manipulation

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = \vec{\nabla}_A\times(\vec{A}\times\vec{B}) + \vec{\nabla}_B\times(\vec{A}\times\vec{B})$$

works as if a regular vector acting only on $A\,(B)$
and apply BAC-CAB rule : $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$.

$$\vec{\nabla}_A\times(\vec{A}\times\vec{B}) = \cancel{\vec{A}(\vec{\nabla}_A\cdot\vec{B})}(\vec{\nabla}_A\cdot\vec{B})\vec{A} - (\vec{\nabla}_A\cdot\vec{A})\vec{B}$$
$$= (\vec{B}\cdot\vec{\nabla}_A)\vec{A} - \vec{B}(\vec{\nabla}_A\cdot\vec{A})$$

Similarly,

$$\vec{\nabla}_B\times(\vec{A}\times\vec{B}) = (\vec{\nabla}_B\cdot\vec{B})\vec{A} - (\vec{\nabla}_B\cdot\vec{A})\vec{B}$$
$$= \vec{A}(\vec{\nabla}_B\cdot\vec{B}) - (\vec{A}\cdot\vec{\nabla}_B)\vec{B}$$

$\therefore$ Now drop $\vec{A}$ and $\vec{B}$ in $\nabla_{A,B}$ and combine two.

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{\nabla}\cdot\vec{A})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - (\vec{A}\cdot\vec{\nabla})\vec{B}.$$

HW Prob. 1.22 (a) & (b)

(a) $\underbrace{(\vec{A}\cdot\vec{\nabla})}_{\text{scalar}}\underbrace{\vec{B}}_{\text{vector}}$; what is the $\hat{x}$-comp ?