Magnetostatics.

$$\vec{F}_q = q(\vec{E} + \vec{v} \times \vec{B})$$

<u>Electrostatics</u>		<u>Magnetostatics</u>	
$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho$	(Gauss's Law)	$\vec{\nabla}\cdot\vec{B} = 0$	(?)
$\vec{\nabla}\times\vec{E} = 0$	(?)	$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$	(Ampere's Law)

Maxwell's Eq for Static Case.

Determine $\vec{E}$ with $\vec{E}\to 0$ as $r\to\infty$ from q.

Determine $\vec{B}$ with $\vec{B}=0$ as $r\to\infty$ from J

<u>EX</u> Consider a copper wire of radius $1\,mm = 10^{-3}\,m$.
Each copper atom provides one free electron.

(i) What is the electron density of Cu?

Atomic mass of Cu $= 64\,gr/mole = 64\times10^{-3}\,Kg/mole$

$N_A = 6\times10^{23}\,/mole$.

ρ_M : mass density of Cu $= 9\,gr/cm^3 = 9\times10^{3}\,kg/m^3$

ρ_e : electron # density of Cu.

$$\rho_e = \frac{(6\times10^{23})}{64\times10^{-3}}\cdot 9\times10^{3} \approx 10^{29}$$

$\therefore$ Charge density $\rho_q = e\times\rho_e = (1.6\times10^{-19})(10^{29}) \approx \underline{2\times10^{10}}\ C/m^3$.

(ii). When $1\,A$ of current is flowing in the wire, what is the drift speed of electrons?

$$J = \frac{I}{\pi a^2} = \frac{1}{\pi(10^{-3})^2} = \frac{1}{\pi}10^{6} \approx 3\times10^{5}\ (A/m^2)$$

$$= \rho_q v$$

$$\therefore\ v = \frac{J}{\rho_q} = \frac{3\times10^{5}}{2\times10^{10}} \approx 1.5\times10^{-5}\ (m/s) = 1.5\times10^{-3}\ cm/s.$$

- Vector Potential $\vec{A}$

$$\vec{\nabla}\times\vec{E}=0 \Rightarrow \vec{E}=-\vec{\nabla}V(\vec{r}) \text{ ——— ①}$$

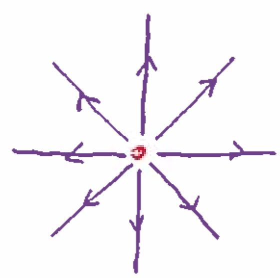
Similarly

$$\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \vec{B}=\vec{\nabla}\times\underbrace{\vec{A}(\vec{r})}_{\text{"vector potential"}} \text{ ——— ②} \qquad \vec{\nabla}\cdot(\vec{\nabla}\times\ )=0.$$

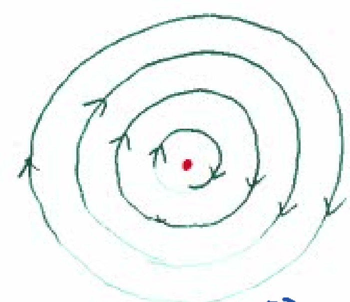
Now we can set up a differential eq. for $\vec{A}$.

$$\vec{\nabla}\times\vec{B}=\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-(\vec{\nabla}\cdot\vec{\nabla})\vec{A}=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\nabla^2\vec{A}$$

$$=\mu_0\vec{J} \text{ ——— ③}$$



Source of $\vec{\nabla}\cdot$ → Charge



Source of $\vec{\nabla}\times$ → current.

In electrostatics, the electric potential possesses an ambiguity since $V(\vec{r})\longrightarrow V(\vec{r})+\text{const}$ produces the same $\vec{E}$:

$$\vec{\nabla}V(\vec{r})=\vec{\nabla}(V(\vec{r})+\text{const})$$

Similarly, in magnetostatics, the vector potential can only be determined with a function $\vec{f}(\vec{r})$ which satisfies

$$\vec{\nabla}\times\vec{f}(\vec{r})=0. \quad \rightarrow \text{Gauge Invariance}$$

$$\therefore \vec{\nabla}\times\vec{A}(\vec{r})=\vec{\nabla}\times(\vec{A}+\vec{f})=\vec{B}$$

If $\vec{f}(\vec{r})=\vec{\nabla}\lambda(\vec{r})$, then $\vec{\nabla}\times\vec{\nabla}\lambda(\vec{r})=0$.

This means we can choose a proper $\lambda(\vec{r})$ to make

$$\vec{\nabla}\cdot\vec{A}=0 \quad \text{in eq ③ above.}$$

Let's say $\vec{\nabla}\cdot\vec{A}_0 \neq 0$. But replacing $\vec{A}_0 \rightarrow \vec{A}_0 + \vec{\nabla}\lambda$ is perfectly fine!

$$\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot(\vec{A}_0 + \vec{\nabla}\lambda) = \vec{\nabla}\cdot\vec{A}_0 + \nabla^2\lambda = 0$$

$$\therefore \boxed{\nabla^2\lambda = -\vec{\nabla}\cdot\vec{A}_0} \longleftrightarrow \left(\nabla^2 V = -\rho/\varepsilon_0 \ : \text{Poisson eq.}\right)$$

$$\therefore \lambda(\vec{r}) = -\frac{1}{4\pi}\int \frac{\vec{\nabla}\cdot\vec{A}_0}{\mathscr{r}}\, d\tau' \qquad V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}}\, d\tau'$$

if $\vec{\nabla}\cdot\vec{A}_0$ is localized!

However if $\vec{\nabla}\cdot\vec{A}_0$ is extended to infinity, then use a different method.

$\boxed{\text{It is always possible to make } \vec{\nabla}\cdot\vec{A} = 0}$ $\overset{\text{eq(2)}}{\Longrightarrow}$ $\boxed{\nabla^2\vec{A} = -\mu_0\vec{J}}$

Vector Poisson eq.

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}\, d\tau'$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}}{\mathscr{r}}\, dl' = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\,'}{\mathscr{r}}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}\,')\, da'}{\mathscr{r}}$$

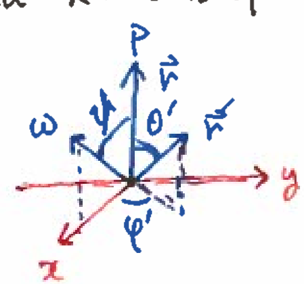
EX

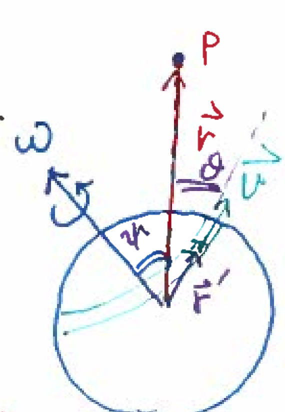
$\vec{K} = \sigma\vec{v}$ and set $\vec{r} = z\hat{z}$

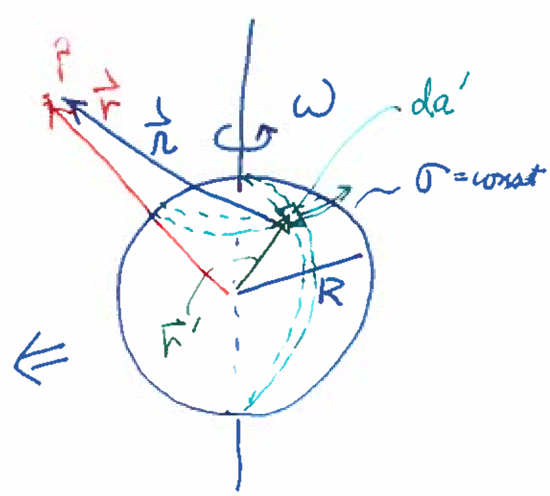
$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

$\mathscr{r} = \sqrt{r^2 + R^2 - 2Rr\cos\theta'}$

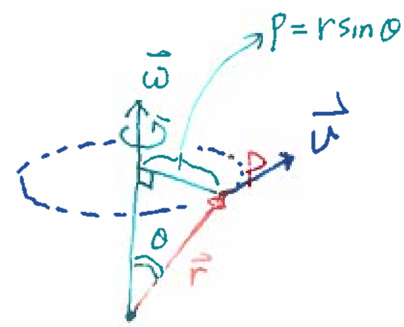
$da' = R^2\sin\theta'\, d\theta'\, d\varphi'$







Digression: Relation between $\vec{\omega}$ and $\vec{v}$



The radius of the circular path of P

$\rho = r\sin\theta$.

$\therefore |\vec{v}| = \rho\dot{\varphi} = (r\sin\theta)\omega$.

The direction is in $\hat{\varphi}$ and consequently in $\hat{\omega}\times\hat{r}$.

Therefore $\vec{v} = \vec{\omega}\times\vec{r}$.

In general if $\hat{e}$ is a unit vector fixed in a rotating body.

$$\frac{d\hat{e}}{dt} = \hat{\omega}\times\hat{e}.$$

$$\vec{v} = \vec{\omega}\times\vec{r}\,' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\varphi' & R\sin\theta'\sin\varphi' & R\cos\theta' \end{vmatrix}$$

$$= R\omega\left[-\cos\psi\sin\theta'\sin\varphi'\,\hat{x} + (\cos\psi\sin\theta'\cos\varphi' - \sin\psi\cos\theta')\hat{y} + \sin\psi\sin\theta'\sin\varphi'\,\hat{z}\right]$$

Since $\int_0^{2\pi}\sin\varphi'\,d\varphi = \int_0^{2\pi}\cos\varphi'\,d\varphi = 0$,

the only surviving term when $\int da'$, is $-\sin\psi\cos\theta'\,\hat{y}$.

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{K}(\vec{r}\,')}{\mathscr{r}}da' = \frac{\mu_0}{4\pi}\sigma(R\omega)\int_0^{2\pi}d\varphi'\int_0^{\pi}d\theta'\,\frac{-\sin\psi\cos\theta'\,R^2\sin\theta'\,d\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\,\hat{y}$$

$$= -\frac{\mu_0 R^3\omega\sigma}{2}\underbrace{\int_0^{\pi}d\theta\,\sin\psi\frac{\cos\theta'\sin\theta'\,d\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}}\,\hat{y}$$

We have done a similar integral in HW and class Prob. 2.7

$$\int_0^{\pi} \frac{\cos\theta' \sin\theta' d\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}} = \int_{-1}^{+1} \frac{t\,dt}{\sqrt{R^2+r^2-2Rrt}} \qquad \left(\begin{array}{l} \cos\theta' = t \\ -\sin\theta' d\theta' = dt \end{array}\right)$$

$$= -\frac{(R^2+r^2+Rrt)}{3R^2r^2}\sqrt{R^2+r^2-2Rrt}\,\Big|_{-1}^{+1}$$

$$= -\frac{1}{3R^2r^2}\left[(R^2+r^2+Rr)|R-r| - (R^2+r^2-Rr)(R+r)\right]$$

$$= \begin{cases} \dfrac{2r}{3R^2} & \text{for } r<R \quad (\text{inside}) \\ \dfrac{2R}{3r^2} & \text{for } r>R \quad (\text{outside}) \end{cases}$$

$$\therefore\ \vec{A}(\vec{r}) = \begin{cases} -\dfrac{\mu_0}{2}R^3\omega\sigma\sin\psi\left(\dfrac{2r}{3R^2}\right)\hat{y} = \dfrac{\mu_0 R\omega\sigma}{3}r(\hat{\omega}\times\hat{r}) & (r<R) \\ -\dfrac{\mu_0}{2}R^3\omega\sigma\sin\psi\left(\dfrac{2R}{3r^2}\right)\hat{y} = \dfrac{\mu_0 R^4\omega\sigma}{r^2}(\hat{\omega}\times\hat{r}) & (r>R). \end{cases}$$

Going back to the original coordinates.

$$\vec{A}(r,\theta,\varphi) = \begin{cases} \dfrac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{r}) = \dfrac{\mu_0 R\omega\sigma}{3} r\sin\theta\,\hat{\varphi} \\ \dfrac{\mu_0 R^4\sigma}{3r^3}(\vec{\omega}\times\vec{r}) = \dfrac{\mu_0 R^4\omega\sigma}{3}\dfrac{\sin\theta}{r^2}\hat{\varphi}. \end{cases}$$

then, $\vec{B} = \vec{\nabla}\times\vec{A}$.

EX $\vec{A}$ of an infinite solenoid.

Consider a line integral on a closed loop of $\vec{A}$

$$\oint \vec{A}\cdot d\vec{l} \underbrace{=}_{\text{Stokes thm}} \int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \int_S \vec{B}\cdot d\vec{a} = \underbrace{\Phi_m}_{\text{Magnetic flux}}$$

$\Phi_e = \int \vec{E}\cdot d\vec{a}$.

$$\therefore \oint_C \vec{A}\cdot d\vec{l} = \Phi_m .$$

The above eq. has the same mathematical structure as Ampere's Law.

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{en} .$$

For a straight current $I = I\hat{z}$, we know that

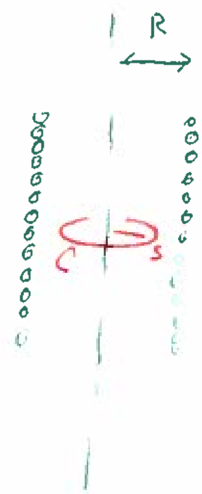
$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\varphi} .$$

Similarly, for a Φ_m in from $\vec{B} = B_0\hat{z}$,

$$\vec{A} = \frac{\Phi_m}{2\pi s}\hat{\varphi} . \quad \text{for } s \text{ outside of confined flux.}$$

The magnetic field inside an infinite solenoid

$$\vec{B} = \mu_0 n I \hat{z} \quad (\text{uniform field}).$$



$\therefore$ For a closed loop inside the solenoid $(s<R)$.

$$\oint_C \vec{A}\cdot d\vec{l} = 2\pi s A_\varphi = \pi s^2(\mu_0 n I) .$$

$$\therefore \vec{A} = \frac{\mu_0 n I}{2} s\,\hat{\varphi} .$$

For $s>R$

$$\oint \vec{A}\cdot d\vec{l} = 2\pi s A_\varphi = \pi R^2(\mu_0 n I)$$

$$\vec{A} = \frac{\mu_0 n I R^2}{s}\hat{\varphi} .$$

$\boxed{\text{HW}}$ 5.23 – 5.27.