Biot-Savart Law.

Const current → Const $\vec{B}$

$\frac{\partial \vec{J}}{\partial t} = 0$ $\quad\quad$ $\frac{\partial \vec{B}}{\partial t} = 0$

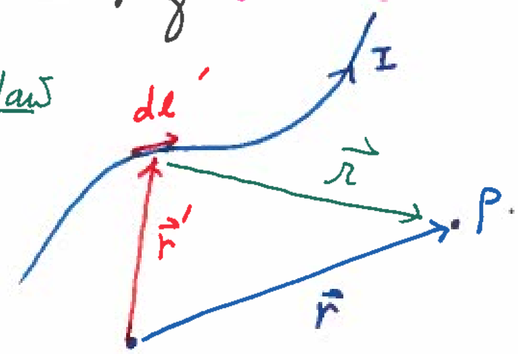
$\vec{J} = \vec{J}(\vec{r})$ $\quad\quad$ $\vec{B} = \vec{B}(\vec{r})$

" Steady State $\frac{\partial}{\partial t} = 0$ including $\frac{\partial \rho}{\partial t} = 0$ "

$\therefore$ $\vec{\nabla}\cdot\vec{J} = 0$ due to Continuity Eq. (for steady current)

$$\boxed{\begin{aligned}\vec{B}(\vec{r}) &= \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,dl' \\ &= \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2}\end{aligned}}$$

Biot-Savart law



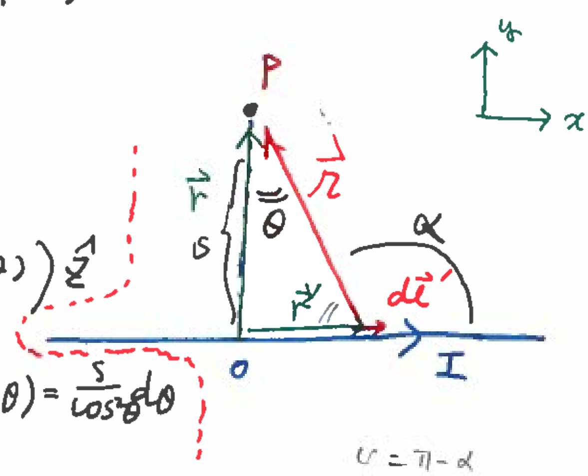
μ_0: vacuum permeability

$\mu_0 = 4\pi \times 10^{-7}$ N/A^2.

$[B] = [\mu_0][I][\frac{1}{L}] = \frac{N}{A^2}\cdot A \cdot \left(\frac{1}{m}\right) = \frac{N}{A\cdot m} = T$.

Ex

$$\vec{B} = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2}.$$



$d\vec{l}'\times\hat{\mathscr{r}} = dl'\sin\alpha = dl'\sin\left(\frac{\pi}{2}-(\pi-\theta)\right)\hat{z}$

$= dr'\cos\theta\ \hat{z}$

Since $\tan\theta = \frac{r'}{s}$ $\Rightarrow$ $dr' = d(s\tan\theta) = \frac{s}{\cos^2\theta}d\theta$

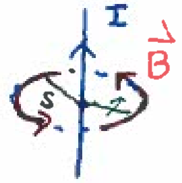
where $\mathscr{r} = \sqrt{r'^2+s^2} = \frac{s}{\cos\theta}$.

$\theta = \pi - \alpha$

$\theta = \frac{\pi}{2}$

$$\therefore \vec{B} = \frac{\mu_0}{4\pi} I \int_{-\infty}^{\pi} \frac{dr'\cos\theta}{\left(\frac{s}{\cos\theta}\right)^2}\hat{z} = \frac{\mu_0}{4\pi} I \int_{-\pi}^{+\pi}\left(\frac{s}{\cos^2\theta}\right)d\theta\,\frac{\cos\theta}{\left(\frac{s}{\cos\theta}\right)^2}\hat{z}$$

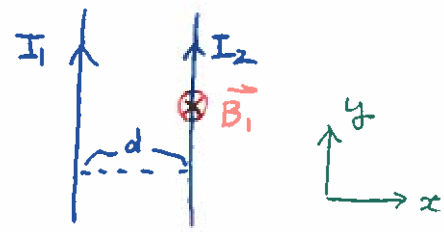
$$= \frac{\mu_0}{4\pi} I \int_{-\pi}^{+\pi} \frac{1}{s}\cos\theta\, d\theta\,\hat{z} = \frac{\mu_0 I}{2\pi s}\hat{z}$$



$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\varphi}$$

- Consider two parallel wires carrying current $\vec{I_1} = I_1\hat{y}$ & $\vec{I_2} = I_2\hat{y}$ ($I_1, I_2 > 0$). At the position of wire 2, the magnetic field from I_1, $\vec{B_1}$, will interact with $\vec{I_2}$ and exert force.

$$\vec{B}_1(d) = -\frac{\mu_0 I_1}{2\pi d}\hat{z}$$



$$\therefore \vec{F}_2 = I_2\int(d\vec{l}_2 \times \vec{B}_1)$$

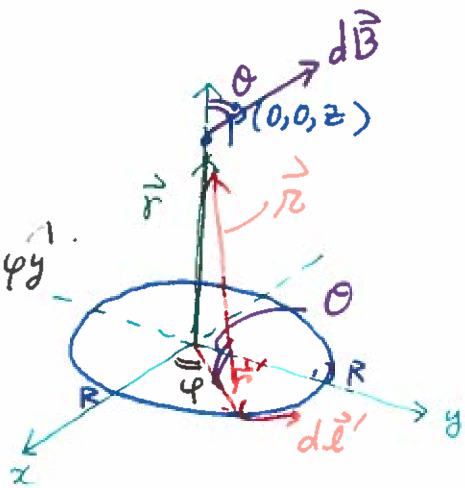
$$= -I_2\frac{\mu_0 I_1}{2\pi d}\int dl_2(\hat{y}\times\hat{z})$$

$$= -I_2\frac{\mu_0 I}{2\pi d}\underbrace{\int dl_2}_{\text{length of wire.}}(\hat{x}) \quad : \text{attractive force}$$

$$\therefore \vec{f}_2 = \vec{F}_2 \Big/ \int dl_2 = -\frac{\mu_0 I_1 I_2}{2\pi d}\hat{x} \quad : \text{force per unit length}$$

$\vec{I}_1\cdot\vec{I}_2 > 0 \Rightarrow$ attractive force

$\vec{I}_1\cdot\vec{I}_2 < 0 \Rightarrow$ repulsive force.



EX

$\vec{r} = z\hat{z}$; $\vec{r}' = R\cos\varphi\,\hat{x} + R\sin\varphi\,\hat{y}$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = z\hat{z} - R\cos\varphi\,\hat{x} - R\sin\varphi\,\hat{y}$

$d\vec{l}' = Rd\varphi\,\hat{\varphi} = Rd\varphi(-\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y})$

$$\therefore d\vec{l}'\times\hat{\mathscr{r}} = \frac{1}{\mathscr{r}}d\vec{l}'\times\vec{\mathscr{r}}$$

$$= \frac{1}{\mathscr{r}}Rd\varphi(-\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y})\times(z\hat{z} - R\cos\varphi\,\hat{x} - R\sin\varphi\,\hat{y})$$

$$= \frac{1}{\mathscr{r}}Rd\varphi\left\{R(\sin^2\varphi + \cos^2\varphi)\hat{z} + z\cos\varphi\,\hat{x} + z\sin\varphi\,\hat{y}\right\}.$$

$$\therefore\ \vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\times\vec{\mathscr{r}}}{\mathscr{r}^3} \qquad \text{where } \mathscr{r}^3 = (z^2+R^2)^{3/2} = \text{const.}$$

$$= \frac{\mu_0 I}{4\pi}\frac{1}{(z^2+d^2)^{3/2}}\int_0^{2\pi} d\varphi \left\{ R^2\hat{z} + \cancel{Rz\sin\varphi\,\hat{y}}^{\,0} + \cancel{Rz\cos\varphi\,\hat{x}}^{\,0} \right\}$$

$$= \frac{\mu_0 I}{2}\frac{R^2}{(z^2+d^2)^{3/2}}\,\hat{z}\,.$$

Or, by symmetry you know only z-comp. survives.

$$\therefore\ B(z) = \frac{\mu_0 I}{4\pi}\int \frac{dl'}{\mathscr{r}^2}\cos\theta \qquad \text{where } \cos\theta = \frac{R}{\mathscr{r}}$$

$$= \frac{\mu_0 I}{4\pi}\frac{R}{\mathscr{r}^3}\int_0^{2\pi} R\,d\varphi = \frac{\mu_0 I}{4\pi}\cdot\frac{2\pi R^2}{\mathscr{r}^3} \Rightarrow \text{same result.}$$

$\boxed{HW}$ 5.9, 5.11, 5.12.

- Divergence and Curl of $\vec{B}$.

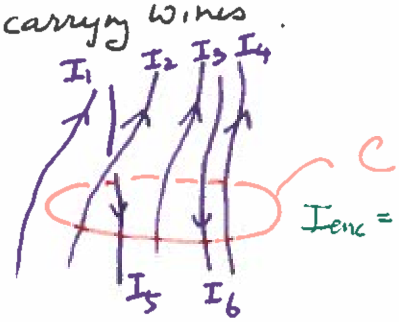
6 Ampere's Law

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

total current enclosed by C

$$I_{enc} = \int \vec{J}\cdot d\vec{a} \qquad \text{for continuous } \vec{J}$$

$$= \sum_{i=1}^{n} I_i \qquad \text{for current carrying wires.}$$



$$I_{enc} = I_2 + I_3 + I_4 - I_5 - I_6$$

In general, $\oint_C \vec{B}\cdot d\vec{l} = \int \vec{J}\cdot d\vec{a}$

$\Downarrow$ Stokes' thm

$$\int (\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \int \vec{J}\cdot d\vec{a}$$

$$\therefore \boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$$: Ampere's law in differential form.

~ Gauss's law.

6 $$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau' \qquad (\vec{\mathscr{r}} = \vec{r}-\vec{r}')$$

$$\vec{\nabla}\cdot\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{\nabla}\cdot\left(\frac{\vec{J}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

$\because \vec{J}(\vec{r}')$

$$\vec{\nabla}\cdot\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\cancelto{0}{(\vec{\nabla}\times\vec{J})} - \vec{J}\cdot\cancelto{0}{\left(\vec{\nabla}\times\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right)}$$

$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$

$= 0$

$$\therefore \boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

EX $\vec{B}$ from a straight wire carrying I.

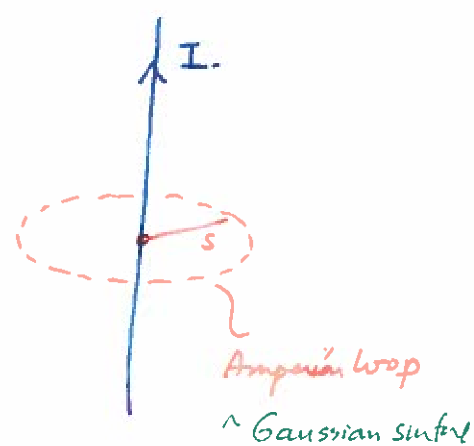
$\downarrow$ Cylindrical Sym

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{en}.$$

$\downarrow$ on the loop. $|\vec{B}| = B = \underline{\underline{\text{const}}}$

$$2\pi s\,B = \mu_0 I$$

$$\therefore \vec{B}(s) = \frac{\mu_0 I}{2\pi s}\hat{\varphi}$$



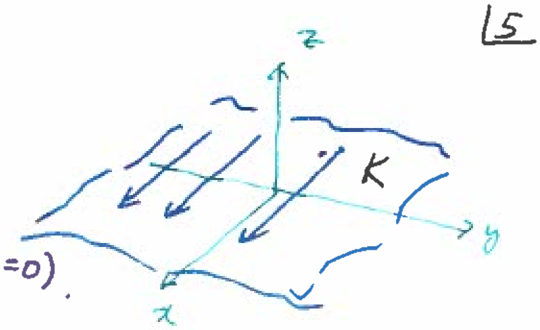
<u>EX</u>

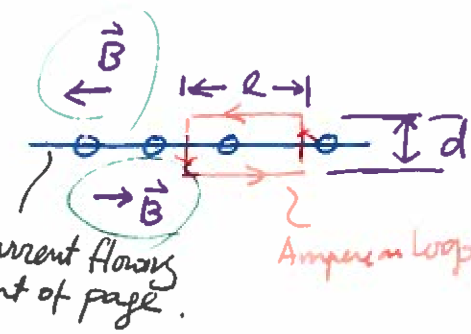
$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{en}.$$

No contribution from the vertical part of the loop ($\because d\vec{l}\cdot\vec{B}=0$).

$$2Bl = \mu_0 K l$$

$$\therefore \vec{B} = \begin{cases} -\frac{\mu_0 K}{2}\hat{y} & (z>0) \\ +\frac{\mu_0 K}{2}\hat{y} & (z<0). \end{cases}$$





<u>EX</u> Solenoid.

For an infinitely long tightly wound solenoid.

$$\vec{B} \sim \hat{z}.$$

n : winding # density

Consider the loop (ii) first. for $L \gg 1$.

$$\oint_{(ii)} \vec{B}\cdot d\vec{l} = \left[-B(a) + B(a+L)\right]l = 0$$

no current.

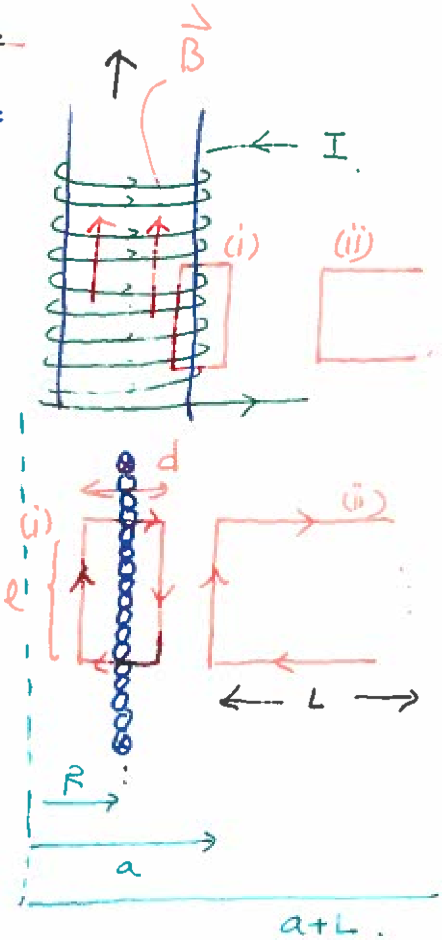
But when $L\to\infty$, $B\to 0$

$\therefore$ $B(a)=0$. (no field outside of solenoid)

Now for loop (i).

$$\oint_{(i)} \vec{B}\cdot d\vec{l} = B_{in}\cdot l - \cancel{B_{out}}^{\,0}\, l = \mu_0 n I l.$$

$$\therefore \vec{B}_{in} = \mu_0 n I\,\hat{z}$$

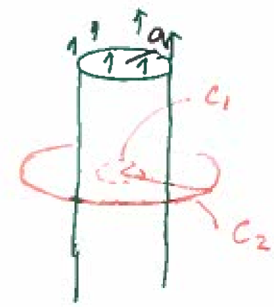


We can position the inside part of the loop anywhere.
Therefore $\vec{B}_{in} = \mu_0 n I\hat{z}$ anywhere inside the infinitely long sol.

$\boxed{HW}$ 5.14, 5.15, 5.17.

5.14.

(a) Cylindrical wire of I and radius a



$B(r<a)$

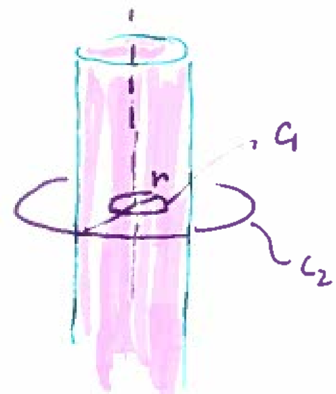
$$\oint_{C_1} \vec{B}\cdot d\vec{l} = \mu_0 I_{en} = 0 \qquad \therefore B(r<a)=0$$

For $r > a$.

$$\oint_{C_2} \vec{B}\cdot d\vec{l} = 2\pi r B_\varphi = \mu_0 I \qquad \therefore B_\varphi = \frac{\mu_0 I}{2\pi r}$$

(b) Solid cylindical wire of uniform current density, of radius r

$$I = \int \vec{J}\cdot d\vec{a}$$



For $r<a$

$$\oint_{C_1} \vec{B}\cdot d\vec{l} = \mu_0 I_{en} = \mu_0 \int \vec{J}\cdot d\vec{a}$$

$$2\pi r B_\varphi = \mu_0 J(\pi r^2)$$

$$\therefore B_\varphi = \frac{\mu_0 J r}{2}$$

For $r>a$

$$2\pi r B_\varphi = \mu_0 J(\pi a^2)$$

$$\therefore B_\varphi = \frac{\mu_0 J a^2}{2r}.$$

(c) For $J = kr$

For $r<a$

$$\underset{\substack{\parallel \\ 2\pi r B_\varphi}}{\oint \vec{B}\cdot d\vec{l}} = \mu_0 I_{en} = \mu_0 \int_0^r (kr')(2\pi r')dr' = \mu_0 \pi k r^3 \left(\frac{2}{3}\right)$$

$$\therefore B_\varphi = \frac{\mu_0 k r^3}{2}\left(\frac{2}{3}\right) = \mu_0 \frac{k r^2}{3}$$

For $r > a$

$$I_{en} = \pi k a^3 \left(\frac{2}{3}\right)$$

$$\therefore B_\varphi = \mu_0 \frac{k a^3}{2r}\left(\frac{2}{3}\right) = \mu_0 \frac{k a^3}{3r}$$