II.16 Boundary Value Problems with Linear Dielectrics (Laplace's Eq). L1

$\vec{P} = \varepsilon_0 \chi_e \vec{E}$; $\quad \vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 (1+\chi_e)\vec{E} = \varepsilon \vec{E} = \varepsilon_0 \varepsilon_r \vec{E}$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\varepsilon_0 \chi_e \vec{\nabla}\cdot\vec{E} = -\frac{\varepsilon_0 \chi_e}{\varepsilon}\vec{\nabla}\cdot\vec{D} = -\frac{\chi_e}{1+\chi_e}\rho_f$$

$\rho_b \propto \rho_f$ in a Linear Homogeneous Isotropic Dielectric Mat.

Therefore, unless free charge is embedded in the dielectric material,

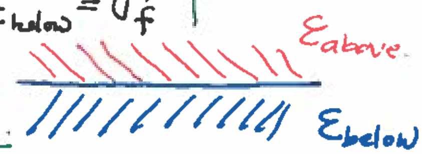
$$\rho = \rho_b + \rho_f = \rho_f \left(1 - \frac{\chi_e}{1+\chi_e}\right) = 0 .$$

And any net charge must reside on the surface (boundary).

$\Rightarrow$ Laplace's Eq $\quad \nabla^2 V = 0$.

$$\boxed{D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f \;;\quad \varepsilon_{above} E^{\perp}_{above} - \varepsilon_{below} E^{\perp}_{below} = \sigma_f}$$

$$\therefore \quad \varepsilon_{above}\frac{\partial V_{above}}{\partial n} - \varepsilon_{below}\frac{\partial V_{below}}{\partial n} = -\sigma_f$$



But always $V_{above} = V_{below}$ at the boundary.

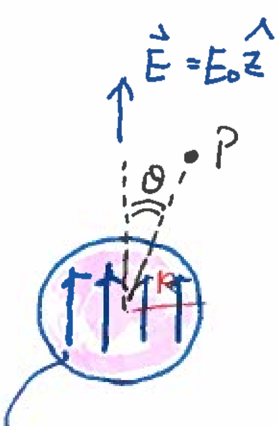
EX Solving $\nabla^2 V = 0$ w/ proper B.C's

(i) $V_{in} = V_{out}$ at $r = R$

(ii) Since $\sigma_f = 0$, $\quad \varepsilon \left.\frac{\partial V_{in}}{\partial r}\right|_R = \varepsilon_0 \left.\frac{\partial V_{out}}{\partial r}\right|_R$.

(iii) $V_{out} \to -E_0 r\cos\theta$ for $r \gg R$

$$\vec{E} = -\vec{\nabla} V = E_0 \hat{z} \cdot \; (z = r\cos\theta)$$



Linear Isotropic Homogeneous D.M. (ε). $\varepsilon_r = \varepsilon/\varepsilon_0$.

For $r < R$

$$V_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta).$$

$B_l = 0$ to have a finite $V_{in}(0,\theta)$.

For $r > R$

$$V_{out}(r,\theta) = -E_0 r\cos\theta + \sum_{l=0}^{\infty} B_l \frac{1}{r^{l+1}} P_l(\cos\theta).$$

B.C. (i).

$$\sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = -E_0 R \underbrace{\cos\theta}_{P_1(\cos\theta)} + \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta).$$

Seperate out $\ell=1$ term, "exclude $\ell=1$"

$$A_1 R^1 P_1(\cos\theta) + \sum_{\ell=0}^{\infty}{}' A_\ell R^\ell P_\ell(\cos\theta) = \left(\frac{B_1}{R^2} - E_0 R\right) P_1(\cos\theta) + \sum_{\ell=0}^{\infty}{}' \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta).$$

$$\begin{cases} A_1 R = \dfrac{B_1}{R^2} - E_0 R & \text{——— ①} \\[2ex] A_\ell R^\ell = \dfrac{B_\ell}{R^{\ell+1}} \quad \text{for } \ell \neq 1 & \text{——— ②} \end{cases}$$

B.C. (ii).

$$\varepsilon \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) = -\varepsilon_0 E_0 \cos\theta - \sum_{\ell=0}^{\infty} \varepsilon_0 \frac{(\ell+1) B_\ell}{R^{\ell+2}} P_\ell(\cos\theta).$$

$$\varepsilon A_1 P_1(\cos\theta) + \sum_{\ell=0}^{\infty}{}' \varepsilon \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) = -\varepsilon_0\left(E_0 + \frac{B_1}{R^3}\right) P_1(\cos\theta) - \sum_{\ell=0}^{\infty}{}' \varepsilon_0 \frac{(\ell+1) B_\ell}{R^{\ell+2}} P_\ell(\cos\theta)$$

$$\varepsilon A_1 = -\varepsilon_0\left(E_0 + \frac{B_1}{R^3}\right) \Rightarrow \varepsilon_r A_1 = -E_0 - \frac{B_1}{R^3} \quad \text{——— ③}.$$

$$\varepsilon_r \ell A_\ell R^{\ell-1} = -\frac{(\ell+1) B_\ell}{R^{\ell+2}} \quad \text{for } \ell \neq 1 \quad \text{——— ④}.$$

* 4 unknows with 4 eq's.

$$A_\ell = B_\ell = 0 \quad \text{for } \ell \neq 1$$

$$A_1 = -\frac{3}{\varepsilon_r + 2} E_0 \quad \text{and} \quad B_1 = \frac{\varepsilon_r - 1}{\varepsilon_r + 2} R^3 E_0.$$

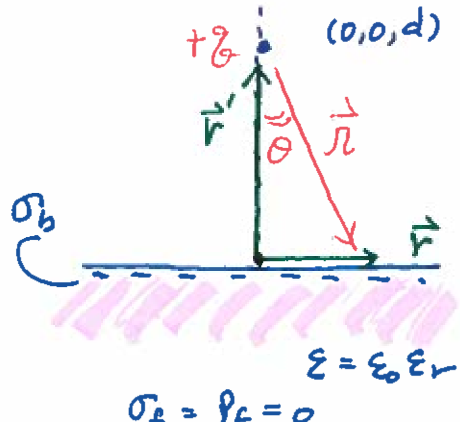
$$\therefore V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

$$= -\frac{3}{\varepsilon_r + 2} E_0 r \cos\theta = -\frac{3}{\varepsilon_r + 2} E_0 z$$

$$\vec{E}_{in} = -\vec{\nabla} V_{in} = -\frac{\partial}{\partial z} V_{in} \hat{z} = \frac{3}{\varepsilon_r + 2} E_0 \hat{z} \quad \text{(uniform field)}.$$

$$V_{out}(r,\theta) = -E_0 r\cos\theta + \frac{\varepsilon_r - 1}{\varepsilon_r + 1} R^3 \frac{E_0 \cos\theta}{r^2}.$$

$$\vec{E}_{out} = -\vec{\nabla} V_{out}.$$



EX Force on $+q$ at $(0,0,d)$

* It is not easy to know the image charge q' and its location in this case.
But we know σ_b <0 will be induced, and the force is attractive.

$$\sigma_b = \vec{P}\cdot\hat{n} = \varepsilon_0 \chi_e \vec{E}_{in}\cdot\hat{n} = \varepsilon_0 \chi_e (E_{in})_z \quad \text{——— (5)}$$

→ This will fully represent the property of the dielectric!

What is $\vec{E}_{in}$? $\vec{E}_{in}$ is generated by $+q$ and σ_b!

$$\vec{E}_{in}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{\mathscr{r}^2}\hat{\mathscr{r}} \underbrace{- \frac{\sigma_b}{2\varepsilon_0}\hat{z}}_{\text{directing to } +\hat{z} \text{ since } \sigma_b < 0}.$$

At the boundary (see figure), the normal component ($\hat{z}$).

$$(E_{in})_z = \frac{-1}{4\pi\varepsilon_0}\cdot\frac{q}{\sqrt{r^2+d^2}}\cos\theta - \frac{\sigma_b}{2\varepsilon_0}. \quad \left(\cos\theta = \frac{d}{\mathscr{r}}\right).$$

$$= -\frac{1}{4\pi\varepsilon_0}\cdot\frac{qd}{(r^2+d^2)^{3/2}} - \frac{\sigma_b}{2\varepsilon_0}. \Rightarrow \text{to Eq (5)}.$$

$$\sigma_b = \varepsilon_0\chi_e(E_{in})_z = -\varepsilon_0\chi_e\left[\frac{1}{4\pi\varepsilon_0}\frac{qd}{(r^2+d^2)^{3/2}} + \frac{\sigma_b}{2\varepsilon_0}\right].$$

Solving for σ_b,

$$\sigma_b = -\frac{1}{2\pi}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{qd}{(r^2+d^2)^{3/2}}.$$

Then the total induced charge q_b.

$$q_b = \int_0^\infty dr\, \sigma_b (2\pi r) dr.$$

$$= -\left(\frac{\chi_e}{\chi_e+2}\right) q d \underbrace{\int_0^\infty \frac{r}{(r^2+d^2)^{3/2}} dr}_{\frac{1}{d}} = -\left(\frac{\chi_e}{\chi_e+2}\right) q$$

If you want to put an image charge pt., $q' = q_b = -\left(\frac{\chi_e}{\chi_e+2}\right) q$.

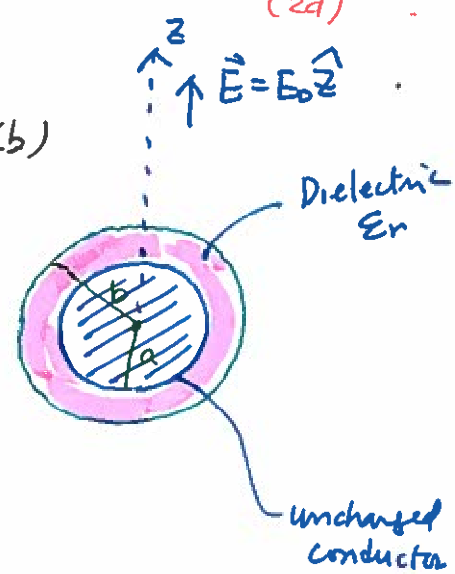
Where to put? At $z = -d$. (not so obvious but it works!)

$$\vec{F}_q = \frac{q}{4\pi\varepsilon_0} \int \frac{\sigma_b(\vec{r}')\, da'}{\mathscr{r}^2} \hat{\mathscr{r}} = -\frac{1}{4\pi\varepsilon_0} \left(\frac{\chi_e}{\chi_e+2}\right) \frac{q^2}{4d^2} \hat{z}.$$

$$4d^2 = (2d)^2.$$

$\boxed{HW}$ 4.24

$$V(r,\theta) = \begin{cases} \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) - E_0 r\cos\theta & (r > b) \\ \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta) & (a < r < b) \\ \text{Const} = 0 & (r < a). \end{cases}$$



B.C.

(i) Continuity at $r = b$.

$$\sum_{l=0}^{\infty} \frac{B_l}{b^{l+1}} P_l(\cos\theta) - E_0 b\cos\theta = \sum_{l=0}^{\infty} \left(A_l b^l + \frac{B_l}{b^{l+1}}\right) P_l(\cos\theta).$$

(ii) Continuity at $r = a$.

$$\sum_{l=0}^{\infty} \left(A_l a^l + \frac{B_l}{a^{l+1}}\right) P_l(\cos\theta) = 0.$$

(iii) $\sigma_f = 0$ at $r = b$.

$$+\varepsilon_0 \left.\frac{\partial V_{r>b}}{\partial r}\right|_b - \varepsilon \left.\frac{\partial V_{a<r<b}}{\partial r}\right|_b = 0.$$

▸ Energy in Dielectric System.

When the whole space where $\vec{E}$ is present is filled w/ a linear dielectric medium with ε, then the resulting electric field can be obtained by replacing $\varepsilon_0 \to \varepsilon$.

Capacitor filled w/ a dielectric material.

$$C = \varepsilon_r C_{vac} = \frac{\varepsilon}{\varepsilon_0} C_{vac}.$$

The energy stored in the capacitor

$$E = W = \frac{1}{2} C V^2 = \frac{1}{2}\left(\frac{\varepsilon}{\varepsilon_0}\right) C_{vac} V^2.$$

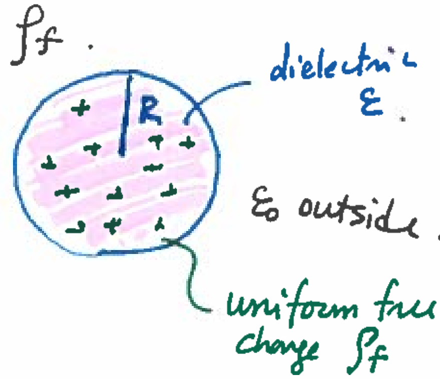
Similarly, the energy stored in field has to be modified accordingly.

$$W = \frac{\varepsilon_0}{2}\int E^2 d\tau. \quad \text{in vacuum}.$$

In a dielectric medium,

$$W = \frac{\varepsilon}{2}\int E^2 d\tau = \frac{1}{2}\int \varepsilon \vec{E}\cdot\vec{E}\, d\tau = \frac{1}{2}\int \vec{D}\cdot\vec{E}\, d\tau.$$

EX Energy of a dielectric sphere of radius R with embedded uniform free charge ρ_f.



$$\vec{\nabla}\cdot\vec{D} = \rho_f.$$

$$\therefore \vec{D}(\vec{r}) = \begin{cases} \dfrac{\rho_f}{3}\vec{r} & (r < R) \\ \dfrac{\rho_f}{3}\dfrac{R^3}{r^2}\hat{r} & (r > R). \end{cases}$$

$$\vec{E}(\vec{r}) = \begin{cases} \dfrac{\rho_f}{3\varepsilon}\vec{r} & (r<R) \\ \dfrac{\rho_f R^3}{3\varepsilon_0}\cdot\dfrac{1}{r^2}\hat{r} & (r>R). \end{cases}$$

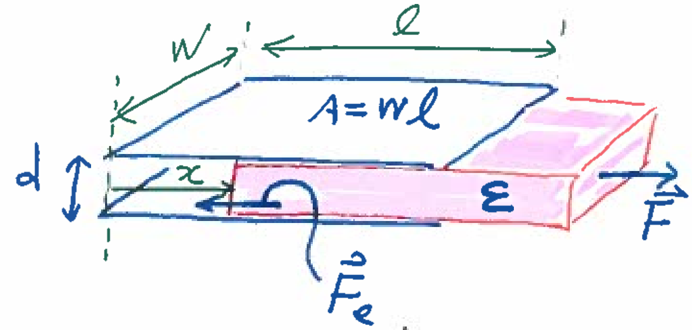
$$\therefore \quad W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\,d\tau$$

$$= \frac{1}{2}\left[\int_0^R \frac{1}{\varepsilon}\left(\frac{\rho_f}{3}\right)^2 r^2\, 4\pi r^2 dr + \int_R^\infty \frac{1}{\varepsilon_0}\left(\frac{\rho_f R^3}{3}\right)^2 \frac{1}{r^4}(4\pi r^2)dr\right]$$

$$= \frac{4\pi}{2\varepsilon_0}\left(\frac{\rho_f}{3}\right)^2\left[\underbrace{\int_0^R \frac{1}{\varepsilon_r} r^4 dr}_{\frac{1}{5\varepsilon_r}R^5} + \underbrace{\int_R^\infty R^6 \cdot \frac{1}{r^2} dr}_{R^5}\right]$$

$$= \frac{2\pi}{9\varepsilon_0}\rho_f^2 R^5\left(\frac{1}{5\varepsilon_r} + 1\right).$$

<u>EX</u>. Pulling the dielectric out of a parallel capacitor.



To pull the dielectric at a constant speed, $|\vec{F}| = |\vec{F}_e|$

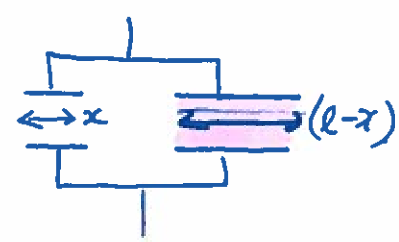
pulling force — electrical force.

$dW = F\,dx$: Work done on the system by the pulling force.

$= -F_e\,dx$.

$$\therefore \quad F_e = -\frac{dW}{dx} = -\frac{d}{dx}\left(\frac{1}{2}CV^2\right)$$

$$C = \frac{\varepsilon_0 w}{d}\left(\varepsilon_r \ell - \chi_e x\right).$$



$$C = \frac{\varepsilon_0 w x}{d} + \frac{\varepsilon w(\ell - x)}{d}$$

$$= \frac{\varepsilon_0 w}{d}\{x + \varepsilon_r(\ell - x)\}$$

$$= \frac{\varepsilon_0 w}{d}\Big(\varepsilon_r \ell + \underbrace{(1-\varepsilon_r)}_{-\chi_e} x\Big)$$

$$\therefore \quad F_e = ?? = -\frac{1}{2}V^2\frac{dC}{dx}.$$

* Can we safely assume V remains const? To be able to fix V = const, there must be external battery which also does work on the system.

To isolate the system we have to establish charges $+Q/-Q$ on the electrodes and isolate.

Then

$$W = \frac{1}{2}\frac{Q^2}{C}$$

$$\therefore \; F_e = -\frac{dW}{dx} = -\frac{d}{dx}\left(\frac{Q^2}{2C}\right) = -\frac{Q^2}{2}\frac{d}{dx}\left(\frac{1}{C}\right) = +\frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx}$$

$$= + \frac{1}{2} V^2 \frac{dC}{dx}$$

complete opposite sign.

$$\therefore \; F_e = -\frac{\varepsilon_0 \chi_e w}{2d} V^2 \quad \text{(pulling in the dielectric)}.$$

$\boxed{HW}$ 4.28.